Sample exam 3
Math 131B, Fall 2021

1. (12 points) Define what it means for a sequence of functions $K_n : \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbf{R}$ to be a Dirac kernel.

In questions 2–4, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

2. (12 points) **TRUE/FALSE:** Let $f \in C^0(S^1)$ be a function such that

$$\hat{f}(n) = \begin{cases} \dfrac{1}{\sqrt{2^n}} & \text{for } n \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then it must be the case that $\displaystyle\int_0^1 |f(x)|^2 \, dx = 1$.

3. (12 points) **TRUE/FALSE:** Let V be an inner product space, and let $\mathcal{B} = \{u_1, \ldots, u_{13}\}$ be an orthogonal set of nonzero vectors in V. Then it is possible that there exist $c_1, \ldots, c_{13} \in \mathbf{C}$ such that

$$\left\| f - \sum_{n=1}^{13} \frac{\langle f, u_n \rangle}{\langle u_n, u_n \rangle} u_n \right\| = 5 \qquad \text{and} \qquad \left\| f - \sum_{n=1}^{13} c_n u_n \right\| = 3.$$

4. (12 points) **TRUE/FALSE:** Let $f, g \in L^2(S^1)$ satisfy the property that $\hat{f}(n) = \hat{g}(n)$ for all $n \in \mathbf{Z}$. Then it is possible that $\|f - g\| = 2$.

5. (12 points) **PROOF QUESTION.** It is a fact (i.e., you may take it as given) that if $F : [0,1] \times [0,1] \to \mathbf{C}$ is a continuous function such that $\dfrac{\partial F}{\partial x}$ is continuous on $[0,1] \times [0,1]$, then for all $x \in [a,b]$,

$$\frac{\partial}{\partial x} \int_0^1 F(x,y) \, dy = \int_0^1 \frac{\partial F}{\partial x} \, dy.$$

In other words, you may switch the order of the operations of differentiation in one variable and integration in the other.

Prove that for $f \in C^1(S^1)$ and $g \in C^0(S^1)$, we have that

$$\frac{d}{dx}((f * g)(x)) = \left(\frac{df}{dx} * g \right)(x).$$

(In other words, prove that the derivative of the convolution $f * g$ is equal to the convolution of $\dfrac{df}{dx}$ and g.)

6. (12 points) **PROOF QUESTION.** Let V be an inner product space, let $\{u_n \mid n \in \mathbf{N}\}$ be an orthogonal set, and suppose that for all $n \in \mathbf{N}$, u_n is orthogonal to g. Prove that for any coefficients $c_n \in \mathbf{C}$, if $\sum_{n=1}^{\infty} c_n u_n$ converges in the inner product metric in V, then

$$\left\langle \sum_{n=1}^{\infty} c_n u_n, g \right\rangle = 0.$$

7. (14 points) **PROOF QUESTION.**

(a) Suppose f and g are continuous on S^1 and $f = g$ almost everywhere in S^1. What can we conclude about f and g?

(b) Suppose that $f \in \mathbf{C}^0(S^1)$ is such that for all $n \neq 0$, we have

$$\left|\hat{f}(n)\right| \leq \frac{1}{|n|^{3/2}}.$$

Prove that the Fourier series of f converges absolutely and uniformly to f.

8. (14 points) **PROOF QUESTION.** Let $\mathcal{H}$ be a Hilbert space, $\{e_n \mid n \in \mathbf{N}\}$ an orthonormal set in $\mathcal{H}$, and $c_n \in \mathbf{C}$.

(a) State the Hilbert Space Absolute Convergence Theorem for $\sum_{n=1}^{\infty} c_n e_n$.

(b) Now suppose that for all $n \in \mathbf{N}$,

$$|c_n| \leq \frac{1}{n^{2/3}}.$$

Prove that $\sum_{n=1}^{\infty} c_n e_n$ converges (in the inner product metric) to some $f \in \mathcal{H}$.

9. (14 points) **PROOF QUESTION.** Let V and W be normed spaces, and let $T : V \to W$ be a function such that $T(0) = 0$ and

$$\|T(f)\| \leq 13 \, \|f\|$$

for all $f \in V$. Prove that T is continuous at 0.