Sample exam 2
Math 131B, Fall 2021

1. (14 points) Let $f : S^1 \to \mathbf{C}$ be given by

$$f(x) = 2x - 1 \qquad \text{for } -\frac{1}{2} \leq x < \frac{1}{2}.$$

Calculate the Fourier coefficients $\hat{f}(n)$ $(n \in \mathbf{Z})$. Show all your work, and do not simplify your final answers.

2. (14 points) Let V be an inner product space.

(a) State the Cauchy-Schwarz inequality for $f, g \in V$.

(b) State the Triangle inequality for $f, g \in V$.

In questions 3–5, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

3. (12 points) **TRUE/FALSE:** It is possible that $f \in C^2(S^1)$ and that $\hat{f}(n) = \dfrac{7}{n}$ for $n \neq 0$.

4. (12 points) **TRUE/FALSE:** Let X be a nonempty open subset of $\mathbf{C}$. If $f_n : X \to \mathbf{C}$ is a sequence of differentiable functions that converges pointwise to some $f : X \to \mathbf{C}$, and $f_n' : X \to \mathbf{C}$ is a sequence of continuous functions that converges uniformly to some $g : X \to \mathbf{C}$, then it must be the case that f is differentiable and $f' = g$.

5. (12 points) **TRUE/FALSE:** If $f_n : [0,1] \to \mathbf{C}$ is a sequence of continuous functions that converges pointwise to some $f : [0,1] \to \mathbf{C}$, then it must be the case that f is continuous.

6. (12 points) **PROOF QUESTION.** Let

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n.$$

(a) Find the radius of convergence R of $f(x)$, with justification. (If you don't remember how to do this, continue to the rest of the problem and just use R as an unknown constant in your answer.)

(b) For which $x \in \mathbf{R}$ is term-by-term differentiation valid?

(c) Use term-by-term differentiation to prove that

$$f'(x) - x f'(x) = 1$$

for all values of x listed in part (b). (Suggestion: You may find the substitution $k = n - 1$ to be useful.)

7. (12 points) **PROOF QUESTION.** Consider the function space $V = C^0([a,b])$ ($a < b$ in $\mathbf{R}$), and define the inner product

$$\langle f(x), g(x) \rangle = \int_a^b f(x)\overline{g(x)}\, dx$$

on V. Suppose $\{p_n(x) \mid n \geq 0\}$ is a set of polynomial functions such that

$$\langle p_i(x), p_j(x) \rangle = \begin{cases} 0 & \text{if } i \neq j, \\ 7 & \text{if } i = j. \end{cases}$$

Now suppose $c_n \in \mathbf{C}$ is a choice of coefficients such that $\sum_{n=0}^{\infty} c_n p_n(x)$ converges absolutely and uniformly to some function $f(x)$. Prove that for $k \geq 0$, we have

$$\int_a^b f(x)\overline{p_k(x)}\, dx = 7c_k.$$

Make sure to justify all steps carefully.

8. (12 points) **PROOF QUESTION.** Let $g_n : S^1 \to \mathbf{C}$ be a sequence of functions such that for $n \geq 1$ and $x \in S^1$, we have that

$$|g_n(x)| \leq \frac{13}{n^{5/2}}.$$

Prove that the series

$$\sum_{n=1}^{\infty} n g_n(x)$$

converges absolutely and uniformly on S^1.