Sample exam 1
Math 131B, Fall 2021

1. (12 points) The goal of this problem is to define the Riemann integral.

Let $v : [a, b] \to \mathbf{R}$ be a bounded real-valued function. Recall that a *partition* P of $[a, b]$ is a finite subset $\{x_0, \ldots, x_n\} \subset [a, b]$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, let $(\Delta x)_i = x_i - x_{i-1}$, and define

$$M(v; P, i) = \sup \{v(x) \mid x \in [x_{i-1}, x_i]\} \qquad m(v; P, i) = \inf \{v(x) \mid x \in [x_{i-1}, x_i]\}$$

$$U(v; P) = \sum_{i=1}^{n} M(v; P, i)(\Delta x)_i \qquad L(v; P) = \sum_{i=1}^{n} m(v; P, i)(\Delta x)_i$$

Let $\mathcal{P}$ be the set of all partitions of $[a, b]$. (I.e., the beginning of the definition of the Riemann integral has been given to you.)

(a) Define the upper and lower Riemann integrals $\overline{\int_a^b} v(x)\, dx$ and $\underline{\int_a^b} v(x)\, dx$.

(b) Define what it means for v to be integrable on $[a, b]$, and define $\int_a^b v(x)\, dx$.

(c) Now let $f : [a, b] \to \mathbf{C}$ be a bounded complex-valued function. Define what it means for f to be integrable on $[a, b]$, and define $\int_a^b f(x)\, dx$.

2. (12 points) Let X be a nonempty subset of $\mathbf{C}$, let $a \in X$, and let $f : X \to \mathbf{C}$ be a function. Give the ϵ-δ definition of what it means for f to be continuous at a.

In questions 3–5, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

3. (12 points) **TRUE/FALSE:** Let $v : [1, 5] \to \mathbf{R}$ be a function such that for every partition P of $[1, 5]$, we have that $U(v; P) - L(v; P) \geq \frac{1}{10}$. Then it is possible that v is integrable on $[1, 5]$.

4. (12 points) **TRUE/FALSE:** There exists a continuous function $f : [1, 3] \to \mathbf{C}$ such that the limit $\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$ does not exist.

5. (12 points) **TRUE/FALSE:** Let S be a nonempty subset of $\mathbf{R}$ such that for all $x \in S$, $x \leq \frac{31}{8}$. Then it is possible that $\sup S = 4$.

6. (12 points) **PROOF QUESTION.** Let $v : [-2, 5] \to \mathbf{R}$ be a bounded function, and suppose that:

- For every $\epsilon > 0$, there exists a partition P_1 of $[-2, 3]$ such that $|7 - L(v; P_1)| < \epsilon$; and

- For every $\epsilon > 0$, there exists a partition P_2 of $[3, 5]$ such that $|6 - L(v; P_2)| < \epsilon$.

Prove that for any $\epsilon > 0$, there exists a partition Q of $[-2, 5]$ such that $|13 - L(v; Q)| < \epsilon$. (Suggestion: You may want to draw a picture of the situation.)

7. (14 points) **PROOF QUESTION.** Let X be a nonempty subset of $\mathbf{C}$, let a be a point of X, and suppose $f : X \to \mathbf{C}$ is a function with the following property:

- For any sequence x_n in X such that $\lim_{n\to\infty} x_n = a$, we have that $\lim_{n\to\infty} f(x_n) = f(a)$.

Let $g : X \to \mathbf{C}$ be defined by $g(x) = (3 + 4i)f(x)$. Use the definition of continuity of g (in any version) to prove that g is continuous at a. You may use the limit laws for sequences, but you may not use the laws of continuity, since the goal of this problem is to re-prove a special case of one of the laws of continuity.

8. (14 points) **PROOF QUESTION.** Let X be a metric space and let L be a point in X.

(a) Let x_n be a sequence in X. Define what it means to say that $\lim_{n\to\infty} x_n = L$.

(b) Now let x_n and y_n be sequences in X such that $\lim_{n\to\infty} y_n = L$ and for all n,

$$d(L, x_n) \leq 17d(L, y_n).$$

Use the definition from part (a) to prove that $\lim_{n\to\infty} x_n = L$. (In particular, do not just quote the Squeeze Lemma, because you are re-proving a special case of the Squeeze Lemma.)