Sample Final Exam
Math 131B, Fall 2020
REVISED Fri Dec 11

Note that problems 1 and 2 are state the definition/theorem problems, which is a type of problem that will not appear on your exam. Conversely, while that precise format of problem no longer makes sense, there may be a problem on your exam that tests roughly the same material, but in a different way.'

Also note that this sample final is taken from last year's final, and last year we covered very different material at the end of the semester (Chs. 9, 10, and parts of 11, instead of Ch. 12). Therefore, a few questions that rely on material we haven't covered have been cut, and there are no questions on this sample that cover Ch. 12. However, this is still better than nothing.

Update: Added a few old questions on the Fourier transform, though not as in-depth as we've done this semester. As always, the exam will be based on our problem sets and similar problems.

1. (14 points) Let f be in $L^2(S^1)$, and N be a positive integer.

(a) Define $f_N(x)$, the Nth Fourier polynomial of f.

(b) Define what it means for $p(x)$ to be a trigonometric polynomial of degree N.

(c) State the *Best Approximation Theorem.* (I.e., what is the most notable property of the Nth Fourier polynomial of f?)

2. (14 points) Suppose $f, g \in C^0(S^1)$.

(a) Define the convolution $(f * g)(x)$.

(b) What is the most notable property of the Fourier coefficients of $f*g$? State the formula precisely.

3. (14 points) Calculate the Fourier coefficients $\hat{f}(n)$ of the function $f : S^1 \to \mathbf{C}$ given for $-\frac{1}{2} \leq x \leq \frac{1}{2}$ by

$$f(x) = \begin{cases} 1 & \text{if } -\frac{1}{4} \leq x \leq \frac{1}{4}, \\ 0 & \text{otherwise.} \end{cases}$$

Show all your work, and do not simplify your final answer.

For questions 4–7, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

4. (13 points) **TRUE/FALSE.** It is possible that there exists a sequence a_n in $\mathbf{R}$ and a continuous function $f : \mathbf{R} \to \mathbf{C}$ such that $\lim_{n\to\infty} a_n = 5$, $f(5) = 13$, and $\lim_{n\to\infty} f(a_n) = 7$.

5. (13 points) **TRUE/FALSE.** If $f : [1, 5] \to \mathbf{C}$ is a Riemann integrable function, then it must be the case that f is continuous.

6. (13 points) **TRUE/FALSE.** Suppose $f \in L^2(S^1)$, and let f_N be the Nth Fourier polynomial of f. Then it must be the case that $\lim_{N\to\infty} \|f - f_N\| = 0$.

7. (13 points) **TRUE/FALSE.** Let f_n be a sequence of continuous functions on $[0,1]$, and suppose that $f : [0,1] \to \mathbf{C}$ is a function such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [0,1]$. Then it must be the case that f is continuous.

8. (16 points) **PROOF QUESTION.** Suppose $f \in C^3(S^1)$. Prove that

$$\sum_{n\in\mathbf{Z}} (2\pi n)\hat{f}(n)$$

converges absolutely. (Suggestion: You may find the Extra Derivative Lemma to be helpful, though it is not necessary for this problem.)

9. (16 points) **PROOF QUESTION.** For $k = 0$ and $k = 1$, define $f_k : S^1 \to \mathbf{C}$ by

$$f_k(x) = \sum_{n\neq 0} \left(\frac{(2\pi i n)^k}{n^3}\right) e_n(x).$$

(a) Prove that if either $k = 0$ or $k = 1$, then $f_k(x)$ converges absolutely and uniformly on S^1.

(b) Prove that $f_0(x)$ is differentiable and $f_0'(x) = f_1(x)$. Be precise about the hypotheses you need to make term-by-term differentiation work.

10. (16 points) **PROOF QUESTION.** Prove that for $f \in C^0(S^1)$ and $n \in \mathbf{Z}$, we have that

$$(e_n * f)(x) = \hat{f}(n)e_n(x).$$

11. (16 points) **PROOF QUESTION.** For $f \in L^2(S^1)$, define $u : S^1 \times (0,+\infty) \to \mathbf{C}$ and $h : (0,+\infty) \to \mathbf{C}$ by

$$u(x,t) = \sum_{n\in\mathbf{Z}} \left(\frac{1}{t^2+1}\right) \hat{f}(n)e_n(x),$$
$$h(t) = \|u(x,t)\|^2,$$

where the norm $\|u(x,t)\|$ is computed in $L^2_x(S^1)$, holding t constant.

(a) Use Parseval to prove that $h(t)$ is equal to a function series in t.

(b) Prove that $h(t)$ converges absolutely and uniformly to a continuous function.

12. (15 points) Let $\mathcal{H}$ be a Hilbert space, and let $\{u_n \mid n \in \mathbf{N}\}$ be a set of nonzero vectors in $\mathcal{H}$.

(a) Define what it means for $\{u_n \mid n \in \mathbf{N}\}$ to be an orthogonal set.

(b) Define what it means for $\{u_n \mid n \in \mathbf{N}\}$ to be an orthogonal basis.

13. (15 points)

(a) For f in the Schwartz space $\mathcal{S}(\mathbf{R})$, define the Fourier transform $\hat{f}(u)$ of f.

(b) State the Fourier inversion theorem in the case of $f \in \mathcal{S}(\mathbf{R})$. (In other words, what do you need to do to $\hat{f}$ to recover f?)

14. (15 points) **PROOF QUESTION.** Suppose $f \in \mathcal{S}(\mathbf{R})$, and let $g(x) = f(-x)$. Prove that for $\gamma \in \mathbf{R}$, $\hat{g}(\gamma) = \hat{f}(-\gamma)$.