Sample Final Exam
Math 131A, Spring 2015

This is the final from Spring 2014, minus one question.

1. (18 points) State the Fundamental Theorems of Calculus, I and II. For simplicity, take each domain, region of continuity, region of differentiability, etc., to be the closed interval $[a, b]$.

2. (13 points) State the Extreme Value Theorem.

3. (15 points) Let $a_n = \dfrac{n^2 + n\cos n}{5^n}$. Determine if the series $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2 + n\cos n}{5^n}$ converges or diverges, and prove your answer.

For questions 4–8, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

4. (13 points) If $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence $R = 2$, then it must be the case that the series $\displaystyle\sum_{n=1}^{\infty} n a_n \left(\frac{5}{3}\right)^{n-1}$ converges.

5. (13 points) Let S be a nonempty subset of $\mathbf{R}$ such that for all $x \in S$, $-13 \leq x \leq 78$. It must be the case that there exists some $m \in S$ such that for all $x \in S$, $m \leq x$.

6. (13 points) For $g : \mathbf{R} \to \mathbf{R}$, it is possible that $\displaystyle\lim_{x \to -3} \frac{g(x) - g(-3)}{x - (-3)} = 5$, $\displaystyle\lim_{n \to \infty} g\left(-3 + \frac{1}{n}\right) = 7$, and $\displaystyle\lim_{n \to \infty} g\left(-3 + \frac{\pi}{n^2}\right) = 5$.

7. (13 points) Let a_n be a sequence such that $\displaystyle\lim_{n \to \infty} a_n = 12$. It is possible that, for all $n \in \mathbf{N}$, $a_{3n} > 13$.

8. (13 points) Let $s_n = \cos(\sin(n^2))$. It must be the case that there exists a increasing sequence of positive integers $n_1 < n_2 < \cdots < n_k < \dots$ such that $\displaystyle\lim_{k \to \infty} s_{n_k}$ converges.

9. (15 points) **PROOF QUESTION.** Define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} x^{4/3} \cos\left(\dfrac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that f is differentiable at 0.

10. (15 points) **PROOF QUESTION.** Define $g : \mathbf{R} \to \mathbf{R}$ by

$$g(x) = \begin{cases} 5x - 15 & \text{if } x \text{ is rational,} \\ x^3 - 17 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that g is **dis**continuous at $x = 2$.

11. (15 points) **PROOF QUESTION.** Let $f : \mathbf{R} \to \mathbf{R}$ be a function (not necessarily continuous, bounded, or any other property) and let $h : \mathbf{R} \to \mathbf{R}$ be a differentiable function such that for all $x \in \mathbf{R}$, $h'(x) = 10 + 7\sin(f(x))$. Prove that if $a, b \in \mathbf{R}$ and $a < b$, then $h(b) - h(a) \geq 3(b - a)$.

12. (15 points) **PROOF QUESTION.** Let

$$S = \left\{ 2 - \frac{3}{n+1} \,\middle|\, n \in \mathbf{N} \right\}.$$

Use the definition of supremum to prove that $\sup S = 2$.

13. (16 points) **PROOF QUESTION.** Let $f_n(x) = \dfrac{1}{n^2 x}$ and let $f(x) = 0$. You make take it as given that f_n converges to f pointwise for all $x \neq 0$ (i.e., you do not need to prove this).

(a) Prove that f_n converges to f uniformly on the interval $\left[\frac{1}{3}, 1\right]$.

(b) Prove that f_n converges to f **non-**uniformly on the interval $(0, 1]$.