Format and topics
Exam 3, Math 131B

General information. Exam 3 will be a timed test of 75 minutes, covering Sects. 7.2–7.4, 7.6, 8.1–8.4, and 8.5.3 of the text, or more importantly, PS07–09. Sect. 7.5 will also be covered, but only to the extent necessary for the material in the other sections; for example, you need to know that $L^2(S^1)$ is a complete inner product space, and that $C^0(S^1) \subset L^2(S^1)$, but you will not be asked any questions that only deal with 7.5. The exam will be cumulative only to the extent that the above sections rely on previous material; for example, you should still know what the supremum of a subset of $\mathbb{R}$ is. However, there will not be any questions on the exam that only cover old material; for example, you will not be asked to define the supremum of a subset of $\mathbb{R}$.

No books, notes, calculators, etc., are allowed. Most of the exam will rely on understanding the problem sets (PS07–09) and the definitions and theorems that lie behind them. If you can do all of the homework, and you know and understand all of the definitions and the statements of all of the theorems we've studied, you should be in good shape.

You should not spend time memorizing proofs of theorems from the book, though understanding those proofs does help you understand the theorems. On the other hand, you should defintely spend time memorizing the *statements* of the important theorems in the text.

Definitions. The most important definitions we have covered are:

7.2	norm	normed space
	norm metric	$\lim_{n\to\infty}$ in a normed space
	bounded (sequence)	continuous (between normed spaces)
	Cauchy	completeness
7.3	orthogonal set	orthonormal set
	generalized Fourier coefficient	projection of f onto $\mathcal{B}$
	$\operatorname{proj}_{\mathcal{B}} f$	orthogonal basis
	orthonormal basis	
7.4	length (of an interval)	countable open cover
	measure zero	almost everywhere
7.5	Lebesgue integrable	$L^1(X)$
	$L^p(X)$	
7.6	Hilbert space	$L^2(X)$
	dense subset (of a metric space)	continuous functions with compact support
8.1	Fourier coefficients in $L^2(S^1)$	Fourier polynomials in $L^2(S^1)$
	Fourier series in $L^2(S^1)$	
8.2	convolution $f * g$	
8.3	Dirac kernel	Dirichlet kernel $D_N(x)$
	Fejér kernel $F_N(x)$	Cesàro sum

Examples. You will also need to be familiar with the key properties of the main examples we have discussed. The most important examples we have seen are:

Sect. 7.2: Normed vector spaces: Inner product spaces, $C^0(X)$ with L^∞ and L^1 norms. Different norms and metrics on $C^0([0,1])$. Incomplete spaces: $C^1([-1,1])$ under L^∞ norm, $C^0([0,2])$ under L^2 norm.

Sect. 7.3: Orthogonal/orthonormal sets: $\{e_n\}$ in $\mathbb{C}^n$, $\{e_n\}$ in $C^0(S^1)$, $\{1, \cos(2\pi nx), \sin(2\pi nx)\}$ in $C^0(S^1)$. Projections: Fourier polynomial, real Fourier polynomial. Orthonormal basis: $\{e_n\}$ in $\mathbb{C}^n$.

Sect. 7.4: Examples of sets of measure zero: finite sets, countable sets, countable unions of measure zero sets. Cantor set is uncountable and measure zero.

Sect. 7.5: Examples of functions that are in $L^2(X)$ for various X (e.g., $f(x) = \dfrac{1}{\sqrt{x}}$ for $X = [0,1]$).

Sect. 7.6: Hilbert spaces: $L^2(X)$, $\ell^2(\mathbb{Z})$.
Sect. 8.3: Dirichlet kernel, Fejér kernel. Fejér kernel as a Dirac kernel, Cesàro sums.

Theorems, results, algorithms. The most important theorems, results, and algorithms we have covered are listed below. You should understand all of these results, and you should be able to state any theorem clearly and precisely. You don't have to memorize theorems by number or page number; however, you should be able to state a theorem, given a reasonable identification of the theorem (either a name, as listed below in **boldface**, or a vague description).

Sect. 7.2: Limit laws in a normed vector space (Thms. 7.2.10 and 7.2.11). Inner products are continuous and therefore series linear/antilinear (Thm. 7.2.13, Cor. 7.2.14).
Sect. 7.3: Best Approximation Theorem (Thm. 7.3.12), especially (3) projection is closest point in span and (4) Bessel's inequality. Always Better Theorem (Cor. 7.3.13).
Sect. 7.4: Countable set has measure zero; countable union of sets of measure zero has measure zero. Set of measure zero cannot contain an open interval. Functions equal a.e. are equal at points of continuity. Lebsegue integral of Riemann integrable function on $\mathbb{R}$ can be computed by an improper (Riemann) integral.
Sect. 7.5: Paraphrase versions of Lebesgue Axioms, especially Lebesgue Axiom 5: $L^2(X)$ complete; Lebesgue Axiom 6: Continuous functions dense in $L^2(X)$.
Sect. 7.6: Completeness of $\ell^2(\mathbb{Z})$. Inner product on $L^2(X)$ is well-defined. Hilbert Space Absolute Convergence Theorem (Thm. 7.6.4). Any generalized Fourier series converges to some element in a Hilbert space (Cor. 7.6.5). Riemann-Lebesgue Lemma (Cor. 7.6.6); Hilbert Space Comparison Test (Cor. 7.6.7). Isomorphism Theorem for Fourier Series (Thm. 7.6.8), including Parseval identities.
Sect. 8.1: List of abstract results applied to $L^2(X)$. Inversion Theorem for Fourier Series (Thm. 8.1.1), Uniform convergence of Fourier Series in $C^1(S^1)$ (Thm. 8.1.2).
Sect. 8.2: Properties of convolutions (Thm. 8.2.4).
Sect. 8.3: Convolution with Dirichlet, Fejér is partial sums (Thm. 8.3.4). Fejér sequence is a Dirac kernel (Thm. 8.3.7).
Sect. 8.4: Convolution with Dirac sequence converges uniformly to identity (Thm. 8.4.1), Cesàro sums converge uniformly (Cor. 8.4.4). $f = g$ if and only if all $\hat{f}(n) = \hat{g}(n)$ (Thm. 8.4.6). Extra Derivative Lemma (Lem. 8.4.7). Term-by-term differentiation of Fourier series (Cor. 8.4.8).

Not on exam. (7.5) L^2 spaces of two-variable functions; continuous functions with compact support. You will not be asked to recite the Lebesgue Axioms, but you should know them at the level of the paraphrase versions at the end of the section. (8.3) Dirac δ "function" (land of wishful thinking).

Other. You should have a working familiarity with the techniques and strategies for proof and logic tips from the proof notes. You do not need to memorize information from the proof notes, but you do need to be able to apply it.

Good luck.