# Math 131B, Mon Nov 09

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: ~~9.4~~ 8.4. Reading for one week from today:
4.8 4.7, 10.1, 12.1.
- PS09 outline due today, full version due in 1 week.
- **NO CLASSES ON WED NOV 11 — VETERANS DAY**
- Problem session, Fri Nov 13, 10:00–noon on Zoom.

# Recap: Two tools

8.2

**Convolutions:**

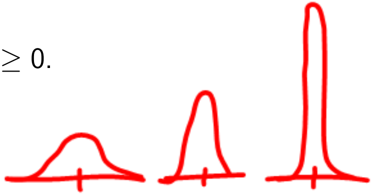
$$(f * g)(x) = \int_0^1 f(x-t)g(t)\,dt.$$

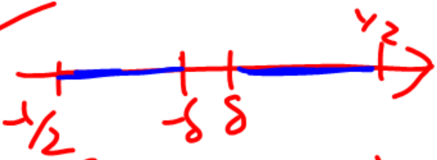
8.3

And **Dirac kernels**, i.e., a sequence of continuous functions $K_n : \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R}$ such that

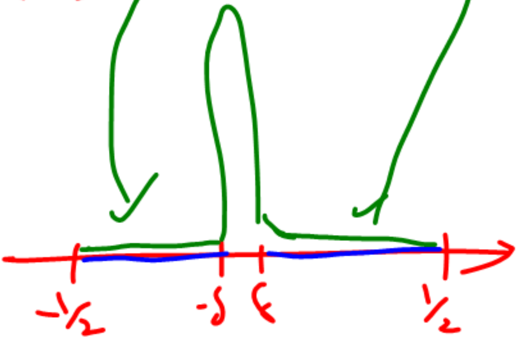
1. For all $n$ and all $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $K_n(x) \geq 0$.
2. For all $n$, $\displaystyle\int_{-1/2}^{1/2} K_n(x)\,dx = 1$; and
3. For any fixed $\delta > 0$, we have

$$\lim_{n\to\infty} \int_{\delta \leq |x| \leq \frac{1}{2}} K_n(x)\,dx = 0.$$






$$\lim_{n\to\infty}\left(\int_{-1/2}^{-\delta} K_n(x)\,dx + \int_{\delta}^{1/2} K_n(x)\,dx\right) = 0.$$




To prove:

(A) $\delta > 0$

(C) $\lim_{n\to\infty}$ —

# The Fejér kernel

8.3

**Example**

The **Dirichlet kernel** $\{D_N \mid N \geq 0\}$ is

(not Dirac)

$$D_N(x) = \sum_{n=-N}^{N} e_n(x).$$

**Example**

The **Fejér kernel** $\{F_N \mid N \geq 1\}$ is

$\boxed{\text{Dirac}}$

$$F_N(x) = \frac{D_0(x) + \cdots + D_{N-1}(x)}{N}.$$

**Theorem**

trig poly degree N

*For $f \in C^0(S^1)$, $f * D_N = f_N$, the $N$th Fourier polynomial of $f$, and*

$$\boxed{s_N(x)} \leftarrow (f * F_N)(x) = \frac{f_0(x) + \cdots + f_{N-1}(x)}{N},$$

trig poly degree N-1

*the average of the Fourier polynomials $f_0, \ldots, f_{N-1}$.*

These conv unif if f cont! (We'll show)

# The goals

$L^2$ **inversion theorem:** If $f \in L^2(S^1)$, then the Fourier series of $f$ converges to $f$ **in the $L^2$ metric.**

$$\lim_{N\to\infty}\int_0^1 |f_N(x)-f(x)|^2\,dx = 0$$ "in $L^2$"

$C^1$ **inversion theorem:** If $f \in C^1(S^1)$, then the Fourier series of $f$ converges **absolutely and uniformly** to $f$.

$f'$ cont on $S^1$

(Prev: If $f \in C^2(S^1)$, $f_N \to$ something ($f$?))

# Main technical result

think: $K_N = F_N$ (Féjer)

**Theorem**

*If $\{K_N\}$ is a Dirac kernel, and $f \in C^0(S^1)$, then*

$$\lim_{N\to\infty} (f * K_N)(x) = f(x),$$

*where convergence is uniform on $S^1$.*

So $\lim_{N\to\infty} \underbrace{s_N(x)}_{\text{trig poly deg } N-1} = f(x)$ unif.

# Two lemmas

ni

To prove the main technical result:

**Lemma** 1

*For any $\epsilon_1 > 0$, there exists some $\delta_1(\epsilon_1) < 1/2$ such that for any $\delta < \delta_1(\epsilon_1)$, any $x \in S^1$, and any $n \in \mathbb{N}$,*

like $|f * K_n - f|$

Inside $(-\delta, \delta]$ →

$$\int_{-\delta}^{\delta} |f(x-t) - f(x)|\,|K_n(t)|\;dt < \epsilon_1.$$

$-\delta \quad \delta$

**Lemma** 2

*For any fixed $\delta > 0$ and $\epsilon_2 > 0$, there exists some $N_2(\delta, \epsilon_2)$ such that for $n > N_2(\delta, \epsilon_2)$ and any $x \in S^1$, we have*

Outside $(-\delta, \delta)$ →

$$\int_{\delta \le |t| \le \frac{1}{2}} |f(x-t) - f(x)|\,|K_n(t)|\;dt < \epsilon_2.$$

We prove the second of these.

(A) $\delta > 0$, $\epsilon_2 > 0$

B/c Dirac, $\lim_{n\to\infty} \int_{\delta \le |t| < \frac{1}{2}} K_n(t)\,dt = 0$

B/c $f$ cont on $S^1$, $f$ bd $\Rightarrow \exists M$ s.t. $|f(x)| \le M$ $\forall x \in S^1$.

So $|f(x-t) - f(x)| \le |f(x-t)| + |f(x)|$
$\le 2M$

So $\forall \epsilon_0 > 0, \exists N_0(\epsilon_0)$ s.t. if $n > N_0(\epsilon_0)$, then

$$\int_{\delta \le |t| < \frac{1}{2}} K_n(t)\,dt < \boxed{\epsilon_0} = \frac{\epsilon_2}{2M}$$

Let $N_2(\epsilon_2) = N_0\left(\frac{\epsilon_2}{2M}\right)$

(A) $n > N_2(\epsilon_2, \delta) = N_0\left(\frac{\epsilon_2}{2M}\right)$

So $\displaystyle\int_{\delta \le |t| \le \frac{1}{2}} K_n(t)\,dt < \frac{\epsilon_2}{2M}$

$$\Rightarrow \int_{\delta \le |t| \le \frac{1}{2}} |f(x-t) - f(x)|\,|K_n(t)|\,dt$$

$$\le \int_{\delta \le |t| \le \frac{1}{2}} 2M K_n(t)\,dt$$

$$= 2M \int_{\delta \le |t| \le \frac{1}{2}} K_n(t)\,dt < 2M\left(\frac{\epsilon_2}{2M}\right)$$

(C) $\displaystyle\int_{\delta \le |t| \le \frac{1}{2}} \underbrace{|f(x-t) - f(x)|}_{\le 2M}\,|K_n(t)|\,dt < \epsilon_2.$

# Proof of main technical result

First observation:

$$f(x) = f(x)\overbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} K_n(t)\,dt}^{=1} = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)K_n(t)\,dt$$

Then combine that with definition of $(f * K_n)(x)$ and two lemmas. . . .

<u>PS09</u>

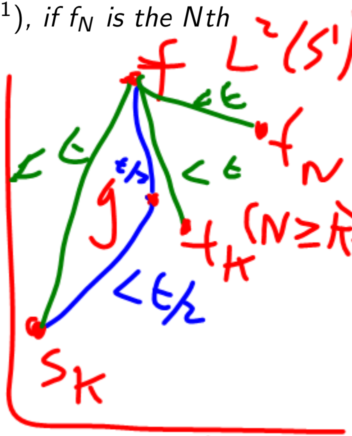
# Proof of the Inversion Theorem

**Inversion Theorem.** *For $f \in L^2(S^1)$, if $f_N$ is the $N$th Fourier polynomial of $f$, then*

$$\lim_{N\to\infty} \|f - f_N\| = 0,$$

*where convergence is in the $L^2$ norm.*

Suppose $f \in L^2(S^1)$ and $\epsilon > 0$.




By LAx 6, $\exists\, g \in C^0(S^1)$ s.t. $\|f - g\| < \frac{\epsilon}{2}$.

tri ineq

B/c $g \in C^0(S^1)$, $S_K$ ($K$ Cesàro sum of $g$) conv unif to $g$, so $\exists\, K$ s.t. $\|g - S_K\| < \frac{\epsilon}{2}$

So $\|f-s_k\| \le \underbrace{\|f-g\|}_{<\epsilon/2} + \underbrace{\|g-s_k\|}_{<\epsilon/2} < \epsilon$

By Best Approx,

$$\|f-f_k\| \le \|f-s_k\| < \epsilon.$$

By Always Better, for $N>k$,

$$\|f-f_N\| \le \|f-f_k\| < \epsilon$$

$$\boxed{\subset} \quad \|f-f_N\| < \epsilon$$

# Extra Derivative Lemma

## Lemma

*If $g \in L^2(S^1)$, then the two-sided series*

$$\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \left( \frac{1}{2\pi n} \right) \hat{g}(n)$$

*converges absolutely (as a series of complex numbers).*

**Proof:**

# $C^1$ uniform convergence

Theorem

*Suppose $f \in C^1(S^1)$. Then the Fourier series of $f$ converges absolutely and uniformly to $f$.*

**Proof:** We first show that the Fourier series of $f$ converges absolutely and uniformly to some $g \in C^0(S^1)$. (This is on PS09.)

Therefore, for all $n \in \mathbb{Z}$, $\hat{g}(n) = \hat{f}(n)$. However, then implies that $f = g$ a.e., and since both $f$ and $g$ are continuous, measure 0 stuff implies that $f = g$ **everywhere**.

# Summary

$L^2$ **inversion theorem:** If $f \in L^2(S^1)$, then the Fourier series of $f$ converges to $f$ **in the** $L^2$ **metric.**

$C^1$ **inversion theorem:** If $f \in C^1(S^1)$, then the Fourier series of $f$ converges **absolutely and uniformly** to $f$.