Math 131B, Mon Nov 02

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 8.1–8.2. Reading for Wed: 8.3.
- PS08 outline due today, full version due Wed.
- Problem session, Fri Nov 06, 10:00–noon on Zoom.

Next week: We meet only on Mon Nov 9; Wed Nov 11 is Veterans Day.
Exam 2 is back; revise errors to recover (1/4) of missing points.

Isomorphism Theorem for (generalized) Fourier Series

= $\mathcal{H}$ Hilbert space, $\mathcal{B} = \{u_n \mid i \in \mathbb{N}\} \subset \mathcal{H}$ orthogonal set of nonzero vectors.

$L^2(S^1)$

IP space, complete in L^2

Theorem

TFAE:

1. *$\mathcal{B}$ is an orthogonal basis for $\mathcal{H}$.*
2. *(Parseval 1) For any $f, g \in \mathcal{H}$, $\langle f, g\rangle = \sum_{n=1}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}\,\langle u_n, u_n\rangle$.*
3. *(Parseval 2) For any $f \in \mathcal{H}$, $\|f\|^2 = \sum_{n=1}^{\infty} \left|\hat{f}(n)\right|^2 \langle u_n, u_n\rangle$.*
4. *For any $f \in \mathcal{H}$, if $\langle f, u_n\rangle = 0$ for all $n \in \mathbb{N}$, then $f = 0$.*

Sp. case: If $\{e_n \mid n \in \mathbb{Z}\}$ ortho*normal* basis for $\mathcal{H}$, then for $f \in \mathcal{H}$,

$e_n(x) = e^{2\pi i n x}$

$$\|f\|^2 = \sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^2.$$

Sp case) $\mathcal{H} = L^2(S^1)$

Once we show $\{e_n \mid n \in \mathbb{Z}\}$ is an orthon. basis, we get:

$$\int_0^1 |f(x)|^2 dx = \|f\|^2 = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2$$

Another interpretation of Isomorphism Theorem for Fourier Series:

This says that any Hilbert space with an orthogonal basis is "isomorphic to" the Hilbert space $\ell^2(\mathbb{Z}) = \{a_n \mid \sum |a_n|^2 < \infty\}$.

Recap: Fundametal defns and facts

8.1

- $L^2(S^1)$ is a Hilbert space. (Lebesgue Axiom 5.)
- Let $\mathcal{B} = \{e_n \mid n \in \mathbb{Z}\}$ in $L^2(S^1)$, where $e_n(x) = e^{2\pi i n x}$. We know that $\mathcal{B}$ is orthonormal.
- For $f \in L^2(S^1)$ and $n \in \mathbb{Z}$, nth Fourier coefficient is:

$$\hat{f}(n) = \langle f, e_n \rangle = \int_0^1 f(x)\, \overline{e_n(x)}\, dx$$

- Nth Fourier polynomial of f is projection of f onto $\{e_{-N}, \ldots, e_N\}$, i.e.,

$$f_N(x) = \sum_{n=-N}^{N} \hat{f}(n) e_n(x).$$

- Fourier series of f is

$$\lim_{N \to \infty} f_N(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n(x).$$

By defn, $\sum_{n\in\mathbb{Z}} \hat{f}(n)e_n(x)$ is sync sum

(Thm in L^2, all conv is absolute, by HS ACT, so sync sum = reg sum)

See Analysis I book, or Appendix A of textbook, for more about the order of summation of a series. Key principle: Absolutely convergent series can be summed in any order with the same result.

Recap: More of what we know

$p(x) = \sum_{n=-N}^{N} c_n e_n(x)$

$\|f_N - f\| \leq \|p - f\|$ whenever p is a trig poly of degree N.

- Best Approximation Theorem: For any $f \in L^2(S^1)$, f_N (the Nth Fourier polynomial of f) is the trigonometric polynomial of degree N that is closest to f in the L^2 metric.
- Always Better Theorem: For $K \leq N$, f_N is closer to f in L^2 than f_K is.
- Bessel's inequality: We always have $\|f_N\| \leq \|f\|$.

Note: While the above results end up being useful, we will actually need to get our hands dirty with ϵ and δ. Coming up. . . .

The main goal now

We want to prove that $\{e_n\}$ is an orthonormal basis for $L^2(S^1)$. More precisely:

Theorem (Inversion Theorem for Fourier Series)

For any $f \in L^2(S^1)$,

$$f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n,$$

where convergence on the RHS, above, is in the L^2 metric.

In terms of pointwise ~~conversion:~~ convergence:

Theorem

If $f \in C^1(S^1)$, then the Fourier series of f converges absolutely and uniformly to f.

These will take some hard work! But also two tools: **Convolution**, and **kernel functions**.

NB! ☆ means: $\left(\sum_{n\in\mathbb{Z}} \hat{f}(n) e_n = \lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}(n) e_n \right)$

$$\lim_{N\to\infty} \left\| \left(\sum_{n=-N}^{N} \hat{f}(n) e_n \right) - f \right\|^2 = 0$$

$$\lim_{N\to\infty} \underbrace{\int_0^1 \left| \left(\underbrace{\sum_{n=-N}^{N} \hat{f}(n) e_n(x)}_{f_N(x)} \right) - f(x) \right|^2 dx}_{\text{avg sq. error}} = 0$$

No 2 (☆☆): Given $A \subset S^1$, A meas. 0

$\exists$ cont $f : S^1 \to \mathbb{C}$ s.t.

Fourier series of f diverges on A.

So can't expect much better than (☆☆).

Convolutions

Back to Riemann integration world for now:

Definition

For $f, g \in C^0(S^1)$, the **convolution** $f * g : S^1 \to \mathbb{C}$ is defined by the formula

const w.r.t. $\int$

$$(f * g)(x) = \int_0^1 f(x - t)g(t)\,dt.$$

sum $= x$

Not obvious that this should be useful! But this turns out to be a kind of product on functions in $C^0(S^1)$.

Properties of convolution

Theorem

For $f, f_i, g, g_i \in C^0(S^1)$ and $c_i \in \mathbb{C}$:

1. $f * g \in C^0(S^1)$. PS08
2. *We have:*

$$(c_1 f_1 + c_2 f_2) * g = c_1(f_1 * g) + c_2(f_2 * g),$$
$$f * (c_1 g_1 + c_2 g_2) = c_1(f * g_1) + c_2(f * g_2).$$

3. $(f * g)(x) = (g * f)(x)$.
4. $((f * g) * h)(x) = (f * (g * h))(x)$.

conv is comm & assoc!

5. *For $f \in C^1(S^1)$, we have $f * g \in C^1(S^1)$ and*
$$\frac{d}{dx}((f * g)(x)) = \left(\frac{df}{dx} * g\right)(x).$$
PS08

I.e., convolution with f transfers the smoothness properties of f to f * g.

Meaning of convolution

Most important property of convolution:

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$$\boxed{\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n).}$$

I.e., convolution corresponds to multiplication of Fourier coefficients. This formula implies that when you combine f and g to form $f * g$:

- f and g reinforce $f * g$ at frequencies they have in common.
- f and g dampen $f * g$ in frequencies where one or more of them have $\hat{f}(n) = 0$.

Application: If the signal f consists of the tones that resonate in some location (say, Cathedral of Notre Dame), the convolution f * g sounds like you played the signal g inside Cathedral of Notre Dame.

Example proof

Thm: $f * g = g * f$.

$$(f * g)(x) = \int_0^1 f(x-t)g(t)\,dt$$

$$(g * f)(x) = \int_0^1 g(x-u)f(u)\,du$$

x const in $\int$

Sub $u = x - t$ $\quad du = -dt$ $\quad t = x - u$

$t=0 \;\; u=x \quad t=1 \;\; u=x-1$

So $(f*g)(x) = \int_x^{x-1} f(u)\,g(x-u)\,(-du)$

$$= \int_{x-1}^{x} f(u)\,g(x-u)\,du$$

Periodicity

$$= \int_0^1 g(x-u) f(u)\,du$$

of f,g on S^1

$$= (g * f)(x)$$

Recall: $h : S^1 \to \mathbb{C}$

$$\int_0^1 h(x)\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} h(x)\,dx = \int_{\pi}^{\pi+1} h(x)\,dx$$

Ex Prove $(f*g)*h = f*(g*h)$.

$$((f*g)*h)(x) = \int_0^1 \Big(\underbrace{\int_0^1 f((x-u)-t)\,g(t)\,dt}_{(f*g)(x-u)}\Big) h(u)\,du$$

$$= \int_0^1 \Big[\int_0^1 \underbrace{f(x-u-t)\,g(t)\,h(u)}_{x,\,u \text{ const}}\,dt\Big]\,du$$

inner

outer

$$\Big\Downarrow \text{ Fubini!}$$

$$= \int_0^1 \int_0^1 f(x-u-t)\, g(t)\, h(u)\, du\, dt$$

$$\Big\Downarrow \text{ subst. } v = u+t$$

$$= \int_0^1 f(x-v) \overbrace{\left(\int_0^1 g(v-u)\, h(u)\, du\right)}^{(g*h)(v)} dv$$

$$= (f*(g*h))(x)$$

Start of 8.2.7

Thm: $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$.

$\widehat{f * g}(n) =$