Math 131B, Wed Oct 28

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 7.6. Reading for Mon: 8.1–8.2.
- PS07 due today.
- Problem session Fri Oct 30, 10:00–noon on Zoom.

The Lebesgue integral so far

Let $X = [a, b]$ or S^1. We can define an integral $\int_X f$ that extends the Riemann integral on X, such that:

- (Lebesgue Axiom 1) Basically any reasonable nonnegative real function can be integrated, though we might get $+\infty$.
- (Lebesgue Axiom 2) If $\int_X |f|$ is finite, then $\int_X f$ exists as a complex number and has the usual properties. of $\int$.
- (Lebesgue Axiom 3) Functions can be changed on a set of measure zero without affecting their integrals.
- (Lebesgue Axiom 4) The Lebesgue integral has the Monotone and Dominated Convergence properties.

$L^2(X)$ as an inner product space $= \{ f \mid \int_X |f|^2 < \infty \}$

Theorem

Let $X = [a, b]$, S^1, or $\mathbb{R}$. Then $L^2(X)$ is a function space, and

square-integrable functions

$$\langle f, g \rangle = \int_X f(x)\overline{g(x)}$$

is an inner product on $L^2(X)$.

Sketch proof:

Given $\int_X |f|^2, \int_X |g|^2 < \infty$,

$$(|f| - |g|)^2 \geq 0$$

$$\Rightarrow |fg| \leq \tfrac{1}{2}|f|^2 + \tfrac{1}{2}|g|^2$$

$$\Rightarrow \int_X |fg| < \infty.$$

$$\Rightarrow |f+g|^2 \le \underbrace{|f|^2 + 2|fg| + |g|^2}_{\text{all } \int_X \text{ finite.}}$$

$$\Rightarrow \int_X |f+g|^2 < \infty$$

$\langle f, g \rangle = \int_X f\bar{g}$ is IP: See PS06 for Riemann version; Lebesgue exactly same.

Lebesgue Axioms 5 and 6

Recall: $C^0(S^1)$ has holes!

Lebesgue Axiom 5: $L^2(X)$ is complete in the L^2 metric. A way to conjure up solutions to problems

Lebesgue Axiom 6: If $X = [a, b]$ or S^1, then $C^0(X)$ is a dense subset of $L^2(X)$. In other words, for every $f \in L^2(X)$ and every $\epsilon > 0$, there exists some $g \in C^0(X)$ with $\|f - g\| < \epsilon$.

Ie, if f_n is Cauchy in $L^2(X)$, then $\lim_{n\to\infty} f_n = f \in L^2(X)$.

<u>Ax 6</u> Given $f \in L^2(X)$, $\epsilon > 0$, $\exists\, g \in C^0(X)$ s.t.

$$\left(\int_X |f-g|^2\right)^{1/2} < \epsilon$$

$$\|f-g\|$$

Recap

Lebesgue Axiom 1: The function space $\mathcal{M}(X)$ contains almost all examples encountered in practice. For any $f \in \mathcal{M}(X)$, $\int_X |f|$ is a well-defined nonnegative extended real number (i.e., the integral could have value $+\infty$).

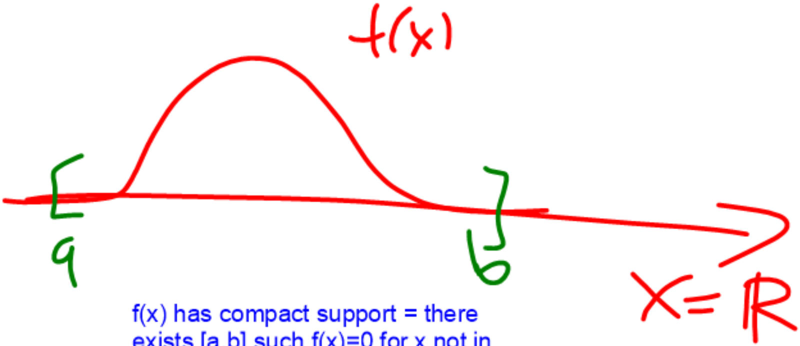
Lebesgue Axiom 2: The Lebesgue integral $\int_X f$ is well-defined on the space $L^1(X)$ of all $f \in \mathcal{M}(X)$ such that $\int_X |f| < \infty$. It extends the Riemann integral and has similar formal properties.

Lebesgue Axiom 3: The Lebesgue integral $\int_X f$ is unaffected by changing the values of f on a set of measure zero.

Lebesgue Axiom 4: Unlike the Riemann integral, the Lebesgue integral satisfies the monotone and dominated convergence properties.

Lebesgue Axiom 5: The function space $L^2(X)$ is an inner product space that is complete in the L^2 metric.

Lebesgue Axiom 6: Continuous functions (or continuous functions with compact support, for $X = \mathbb{R}$) are dense in $L^2(X)$.



f(x) has compact support = there exists [a,b] such f(x)=0 for x not in [a,b].

Hilbert spaces

See also: Statistics, machine learning....

Definition

A **Hilbert space** is an inner product space that is complete in the inner product metric. (L^2 metric)

THE example: By Lebesgue Axiom 5, $L^2(S^1)$ is a Hilbert space. (This is the only reason we need Lebesgue!)

(and so is $L^2(\mathbb{R})$)

Goal of 7.6

$, L^2(S^1)$

Recall: To say $\{e_n \mid n \in \mathbb{Z}\}$ is an orthonormal basis for $\mathcal{H}$ means that $\{e_n \mid n \in \mathbb{Z}\}$ is orthonormal and that for $f \in \mathcal{H}$, we have that

$$f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n,$$

RHS converges AND is equal to f.

where **convergence is in** L^2, i.e.,

$$\lim_{N \to \infty} \left\| f - \sum_{n=-N}^{N} \hat{f}(n) e_n \right\| = 0.$$

Goal of 7.6: If $\mathcal{H}$ is a Hilbert space with an orthonormal basis, what can we say about $\mathcal{H}$?

(Not yet ready to prove that $\{e_n \mid n \in \mathbb{Z}\}$ is an orthonormal basis for $\mathcal{H}$.)

Hilbert Space Absolute Convergence Theorem

$\mathcal{H}$ Hilbert space, $\mathcal{B} = \{u_n \mid n \in \mathbb{N}\}$ an orthogonal set of nonzero vectors in $\mathcal{H}$, $c_n \in \mathbb{C}$. (Everything we do also works for $\mathcal{B} = \{u_n \mid n \in \mathbb{Z}\}$, but we stick with $\mathbb{N}$ to avoid saying everything twice.)

One cool thing about Hilbert spaces:

Theorem (HSACT)

TFAE:

1. $\sum_{n=1}^{\infty} c_n u_n$ *converges to some element of* $\mathcal{H}$. accio!

   in L^2 metric

2. $\sum_{n=1}^{\infty} |c_n|^2 \|u_n\|^2$ *converges in* $\mathbb{R}$.

   sum of the squared lengths of vectors c_n u_n

Proof: PS08. This is the **only** place we use completeness of $\mathcal{H}$, so the only place we (very indirectly) need Lebesgue!

In a Hilbert space, generalized Fourier series all converge

$\mathcal{H}$ Hilbert space, $\mathcal{B} = \{u_n \mid n \in \mathbb{N}\}$ an orthogonal set of nonzero vectors in $\mathcal{H}$, $\cancel{c_n \in \mathbb{C}}$.

Corollary

The generalized Fourier series of any $f \in \mathcal{H}$ relative to $\mathcal{B}$ converges to some element of $\mathcal{H}$ (though not necessarily f).

Proof: By Bessel (Sec 7.3), we have

$$\cancel{\sum_{n=1}^{N} |c_n|^2 \|u_n\|^2 \leq \|f\|^2}$$

~~for any N.~~

(Bessel) $\displaystyle\sum_{n=1}^{\infty} |\hat{f}(n)|^2 \|u_n\|^2 \leq \|f\|^2$

So seq $\displaystyle\sum_{n=1}^{N} |\hat{f}(n)|^2 \|u_n\|^2$ (seq in N)

is an increasing seq, bd above.

So $\sum_{n=1}^{N} |\hat{f}(n)|^2 \|u_n\|^2$ convs (to its sup)

$\Rightarrow \sum_{n=1}^{\infty} |\hat{f}(n)|^2 \|u_n\|^2$ convs.

$\Rightarrow \sum_{n=1}^{\infty} \hat{f}(n) u_n$ convs by HSACT.

Special case: For any f in L^2(S^1), we see that the Fourier Series converges in the L^2 metric.

(This is subtle: There exist continuous functions on S^1 whose Fourier Series diverge on an uncountable set, or in fact, diverge on any set of measure 0 in S^1.)

Hilbert Space Comparison Test

Corollary

$\mathcal{H}$ Hilbert space, $\mathcal{B} = \{u_n \mid n \in \mathbb{N}\}$ an orthogonal set of nonzero vectors in $\mathcal{H}$, $b_n, c_n \in \mathbb{C}$. If $\displaystyle\sum_{n=1}^{\infty} c_n u_n$ converges in $\mathcal{H}$, and $|b_n| \leq |c_n|$ for all $n \in \mathbb{N}$, then $\displaystyle\sum_{n=1}^{\infty} b_n u_n$ also converges in $\mathcal{H}$.

Proof:

$$\sum_{n=1}^{\infty} c_n u_n \text{ conv} \Rightarrow \sum_{n=1}^{\infty} |c_n|^2 \|u_n\|^2 \text{ convs} \quad \text{(HSACT)}$$

$$\Rightarrow \sum_{n=1}^{\infty} |b_n|^2 \|u_n\|^2 \text{ convs} \quad \text{(comparison)}$$

$$\Rightarrow \sum_{n=1}^{\infty} b_n u_n \text{ convs} \quad \text{(HSACT)}$$

Isomorphism Theorem for Fourier Series

$\mathcal{H}$ Hilbert space, $\mathcal{B} = \{u_n \mid i \in \mathbb{N}\} \subset \mathcal{H}$ orthogonal set of nonzero vectors.

Theorem

TFAE:

I.e., < , > in H is just the dot product, computed with respect to coords in the basis B.

1. *$\mathcal{B}$ is an orthogonal basis for $\mathcal{H}$.*
2. *(Parseval 1) For any $f, g \in \mathcal{H}$, $\langle f, g\rangle = \sum_{n=1}^{\infty} \hat{f}(n)\overline{\hat{g}(n)} \langle u_n, u_n\rangle$.*
3. *(Parseval 2) For any $f \in \mathcal{H}$, $\|f\|^2 = \sum_{n=1}^{\infty} \left|\hat{f}(n)\right|^2 \langle u_n, u_n\rangle$.*
4. *For any $f \in \mathcal{H}$, if $\langle f, u_n\rangle = 0$ for all $n \in \mathbb{N}$, then $f = 0$.*

Sp. case: If $\{e_n \mid n \in \mathbb{Z}\}$ ortho*normal* basis for $\mathcal{H}$, then for $f \in \mathcal{H}$, $L^2(S^1)$

$$\int_0^1 |f(x)|^2 dx = \|f\|^2 = \sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^2 .$$

Proof of (1) $\Rightarrow$ (2)

Suppose $\mathcal{B}$ is an orthogonal basis for $\mathcal{H}$. By defn basis,

$$f = \sum_{n=1}^{\infty} \hat{f}(n) u_n, \quad g = \sum_{k=1}^{\infty} \hat{g}(k) u_k .$$

$$\Rightarrow \langle f, g \rangle = \left\langle \sum_{n=1}^{\infty} \hat{f}(n) u_n, \sum_{k=1}^{\infty} \hat{g}(k) u_k \right\rangle$$

cont of IP

$$= \sum_{n=1}^{\infty} \hat{f}(n) \left(\sum_{k=1}^{\infty} \overline{\hat{g}(k)} \langle u_n, u_k \rangle \right)$$

So this sum collapses to $k=n$ term

$= 0$ unless $n = k$

$$= \sum_{n=1}^{\infty} \hat{f}(n)\, \overline{\hat{g}(n)} \langle u_n, u_n \rangle .$$

What we know and don't know

- We know that $L^2(S^1)$ is a Hilbert space. (Rather, we essentially assume this by Lebesgue Axiom 5.)
- We therefore know that if we can show that $\{e_n \mid n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(S^1)$, all kinds of good stuff happens.
- We **don't** yet know that $\{e_n\}$ is an orthonormal basis for $L^2(S^1)$! (In fact, this is just a restatement of our main problem, but in L^2.) So that's our main job now.