Math 131B, Mon Oct 26

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 7.5. Reading for ~~Mon~~: 7.6. Wed
- PS07 outline due today, full version due Wed.
- Problem session Fri Oct 30, 10:00–noon on Zoom.

Review of measure zero

Definition

To say that $E \subseteq \mathbb{R}$ has **measure zero** means that for any $\epsilon > 0$, there exists some open cover $\{U_i\}$ of E such that $\displaystyle\sum_{i=1}^{\infty} \ell(U_i) < \epsilon$.

Example. If $E = \{x_i\}$ is a countable subset of $\mathbb{R}$, then E has measure zero.

PS07: If each E_i has measure zero, then $\displaystyle\bigcup_{i=1}^{\infty} E_i$ has measure zero.

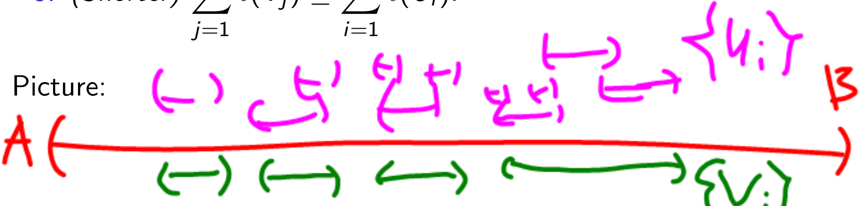
A technical lemma

Lemma

(A, B) open int in $\mathbb{R}$, $\{U_i\}$ countable collection of open int in (A, B). There exists a countable collection $\{V_j\}$ of bounded open ints s.t.:

1. *(Disjoint) For $j \neq k$, $V_j \cap V_k = \emptyset$;*
2. *(Union) $\displaystyle\bigcup_{j=1}^{\infty} V_j = \bigcup_{i=1}^{\infty} U_i$; and*
3. *(Shorter) $\displaystyle\sum_{j=1}^{\infty} \ell(V_j) \leq \sum_{i=1}^{\infty} \ell(U_i)$.*

Picture:



Typical of intro measure theory: "Obvious", but proof is complicated.

A set of measure zero can't contain an interval

Theorem

If E is a set of measure zero, and (a, b) is any open interval in $\mathbb{R}$, then (a, b) is not contained in E.

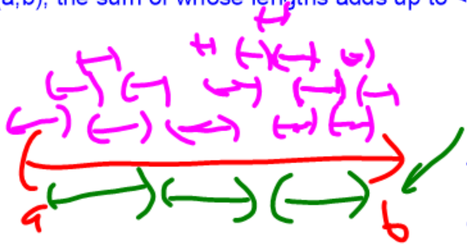
Defn of m.0

Proof:

By contradiction, suppose E has measure 0, and (a,b) contained in E.

Choose an open cover of E, the sum of whose lengths add up to < (b-a)/2.

If you only consider the parts of that open cover that are in (a,b), we get an open cover of (a,b), the sum of whose lengths adds up to < (b-a)/2.



Either this green cover (from Lemma) must be (a,b), contradiction; or there's at least one point missing, contradiction.

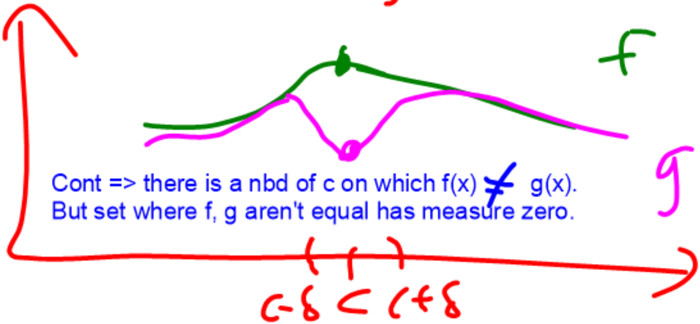
Continuous and equal a.e. means equal

The set of x s.t. f(x) is not equal to g(x) has measure zero.

Corollary

Suppose $X = [a, b]$ or $\mathbb{R}$ and for some $f, g : X \to \mathbb{C}$, we have that $f(x) = g(x)$ for almost all $x \in X$. Then for $c \in X$, if f and g are continuous at c, then $f(c) = g(c)$.

Proof: ABC $f(c) \neq g(c)$



Cont => there is a nbd of c on which f(x) $\neq$ g(x).
But set where f, g aren't equal has measure zero.

The Lebesgue integral

$\boxed{4.5}$

Let $X = [a, b]$ or S^1.
We have seen already that $C^0(X)$ has "holes" in it w.r.t. the L^2 metric. We will **assume** that there exists a larger space of functions with the following properties (roughly speaking):
Lebesgue Axiom 1: The function space $\mathcal{M}(X)$ contains almost all examples encountered in practice. For any $f \in \mathcal{M}(X)$, $\int_X |f|$ is a well-defined nonnegative extended real number (i.e., the integral could have value $+\infty$).
Lebesgue Axiom 2: The Lebesgue integral $\int_X f$ is well-defined on the space $L^1(X)$ of all $f \in \mathcal{M}(X)$ such that $\int_X |f| < \infty$. It extends the Riemann integral and has similar formal properties.
Lebesgue Axiom 3: The Lebesgue integral $\int_X f$ is unaffected by changing the values of f on a set of measure zero.

Lebesgue Axiom 4: Unlike the Riemann integral, the Lebesgue integral satisfies the monotone and dominated convergence properties.

to be explained

Lebesgue Axiom 5: The function space $L^2(X)$ is an inner product space that is complete in the L^2 metric.

Lebesgue Axiom 6: Continuous functions (or continuous functions with compact support, for $X = \mathbb{R}$) are dense in $L^2(X)$.

to be reviewed/explained later

$$\|f\|^2 = \int_X |f|^2$$

$$d(f,g) = \|f-g\|$$

Lebesgue Axiom 1

$X = [a,b)$ or $\mathbb{R}$

$\mathcal{M}(X)$ is the space of **measurable functions** on X and:'

1. (Riemann integrable implies measurable) If $f : X \to \mathbb{C}$ is Riemann integrable on every closed and bounded subinterval of X, then f is measurable.
2. (Closed under algebraic operations) If $f, g \in \mathcal{M}(X)$, then $f(x)g(x)$, $\overline{f(x)}$, and $|f(x)|$ are measurable (and also $f(x) + g(x)$ and $cf(x)$ for $c \in \mathbb{C}$).
3. (Closed under limits) If $f(x) = \lim_{n\to\infty} f_n(x)$ and each $f_n \in \mathcal{M}(X)$, then $f \in \mathcal{M}(X)$.

   very different!

4. (Nonnegative integral) If $f \in \mathcal{M}(X)$ and f is real-valued and nonnegative, then there exists $\displaystyle\int_X f$ in $\mathbb{R} \cup \{+\infty\}$ called the **Lebesgue integral of f on X**.
5. (Monotonicity) For real-valued nonnegative $f, g \in \mathcal{M}(X)$, if $f(x) \le g(x)$ for all $x \in X$, then $\displaystyle\int_X f \le \int_X g$.

Lebesgue integrable functions

Concrete way of looking at integral of measurable function: we include certain kinds of improper integrals.

$f(x) = \frac{1}{x^2}$

$\int_{(0,1]} f = +\infty$

Definition

To say that $f : X \to \mathbb{C}$ is **Lebesgue integrable**, or simply **integrable**, means that $f \in \mathcal{M}(X)$ and $\int_X |f|$ is finite. We also define $L^1(X)$ to be the set of all Lebesgue integrable functions on X. More generally, for finite $p \geq 1$, we define $L^p(X)$ to be the set of all $f \in \mathcal{M}(X)$ such that $\int_X |f|^p$ is finite.

$L^2(X) = \{ f \mid \int_X |f|^2 < \infty \}$

not Leb. int

not in $L^2((0,1])$

Lebesgue Axiom 2

$L^1(X)$ subspace, and for $f \in L^1(X)$, there exists **Lebesgue integral** $\displaystyle\int_X f$ s.t.:

1. (Extends nonnegative integral) If $f \in L^1(X)$ and f is real and nonnegative, then $\displaystyle\int_X f$ has the same value as before.
2. (Extends Riemann integral) If $f : X \to \mathbb{C}$ is Riemann integrable on some $[a, b] \subseteq X$ and $f(x) = 0$ for all $x \notin [a, b]$, then $f \in L^1(X)$ and $\displaystyle\int_X f = \int_a^b f(x)\,dx$.
3. (Linearity) If $f, g \in L^1(X)$ and $a, b \in \mathbb{C}$, then $\displaystyle\int_X af + bg = a\int_X f + b\int_X g$.
4. (Additivity of domain) If $X = [a, b]$, $Y = [b, c]$, and $Z = [a, c]$, and $f \in L^1(X)$ and $f \in L^1(Y)$ when restricted to those domains, then $\displaystyle\int_Z f = \int_X f + \int_Y f$.

Also:

(Conjugates and absolute value) If $f \in L^1(X)$, then $\int_X \overline{f} = \overline{\int_X f}$ and $\left|\int_X f\right| \leq \int_X |f|$.

(triangle inequality for integrals)

Upshot of Axiom 2: We can do computations with Lebesgue integral as if it were Riemann integral.

Lebesgue Axiom 3

For any $f \in \mathcal{M}(X)$, we have following properties of $\int_X f$, either in the complex sense or the nonnegative sense.

1. (Up to measure zero) If $f = g$ almost everywhere in X, then g is also measurable; and if we also have that $f \in L^1(X)$, then $g \in L^1(X)$ and $\int_X f = \int_X g$. In other words, $\int_X f$ is "only defined up to sets of measure zero."
2. (Zero integral of nonnegative implies zero a.e.) If f is real-valued and nonnegative and $\int_X f = 0$, then $f = 0$ almost everywhere in X.

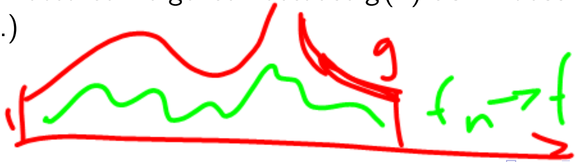
Ex. $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ $\int_{[0,1]} f = 0$

Lebesgue Axiom 4

Compare the two NOs having to do with integrals.

Let $f_n : X \to \mathbb{C}$ be a sequence in $\mathcal{M}(X)$, and let $f : X \to \mathbb{C}$ be a function such that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. in X.

1. (Monotone convergence) If the f_n are nonnegative real measurable and $f_n(x) \le f_{n+1}(x)$ for all $x \in X$, then $\int_X f = \lim_{n\to\infty} \int_X f_n$.
2. (Dominated convergence) If there exists some Lebesgue integrable $g : X \to \mathbb{R}$ such that $|f_n(x)| \le g(x)$ for all $n \in \mathbb{N}$ and $x \in X$, then $\int_X f = \lim_{n\to\infty} \int_X f_n$. (This is called "dominated convergence" because $g(x)$ **dominates** each $f_n(x)$.)



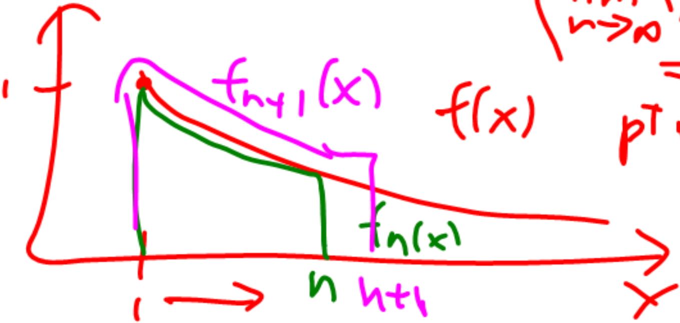
Example

Let $X = [1, +\infty]$, and let $f, f_n : X \to \mathbb{R}$ $(n \in \mathbb{N})$ be defined by

$$f(x) = \frac{1}{x^2}, \qquad f_n(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \leq n, \\ 0 & \text{if } x > n. \end{cases}$$

Prove that f is measurable, f is Lebesgue integrable on X, and $\int_X f = 1$.

$$\left(\lim_{n\to\infty} f_n(x) = f(x) \text{ pt wise}\right)$$



f measurable: Each f_n(x) is Riemann integrable on any finite interval, so each f_n is measurable (Ax 1). f is the limit of measurable functions, so f is measurable (Ax 1).

Note: $f_{n+1}(x) \geq f_n(x)$, so use Mono Conv (Ax 4). So:

$$\int_X f = \lim_{n\to\infty} \int_1^n \frac{1}{x^2}$$

Leb=Rie

$$= \lim_{n\to\infty} \int_1^n x^{-2}\,dx$$

$$= \lim_{n\to\infty} -x^{-1}\Big]_1^n = \lim_{n\to\infty}\left(-\frac{1}{n}+1\right) = 1.$$

$L^2(X)$ as an inner product space

Theorem

Let $X = [a, b]$, S^1, or $\mathbb{R}$. Then $L^2(X)$ is a function space, and

$$\langle f, g \rangle = \int_X f(x)\overline{g(x)}$$

is an inner product on $L^2(X)$.

Lebesgue Axioms 5 and 6

Lebesgue Axiom 5: $L^2(X)$ is complete in the L^2 metric.

Lebesgue Axiom 6: If $X = [a, b]$ or S^1, then $C^0(X)$ is a dense subset of $L^2(X)$. In other words, for every $f \in L^2(X)$ and every $\epsilon > 0$, there exists some $g \in C^0(X)$ with $\|f - g\| < \epsilon$.

Recap

Lebesgue Axiom 1: The function space $\mathcal{M}(X)$ contains almost all examples encountered in practice. For any $f \in \mathcal{M}(X)$, $\displaystyle\int_X |f|$ is a well-defined nonnegative extended real number (i.e., the integral could have value $+\infty$).

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