Math 131B, ~~Wed Sep 30~~ Mon Oct 5

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 5.3, 6.1–6.2. Reading for Wed: 6.4, 7.1.
- **New deadlines:** PS05 due Wed; outline for PS06 due Fri.
- Problem session Fri Oct 09, 10:00–noon on Zoom.

Outlines for PS05 accepted tonight.

Exam 2 in two weeks.

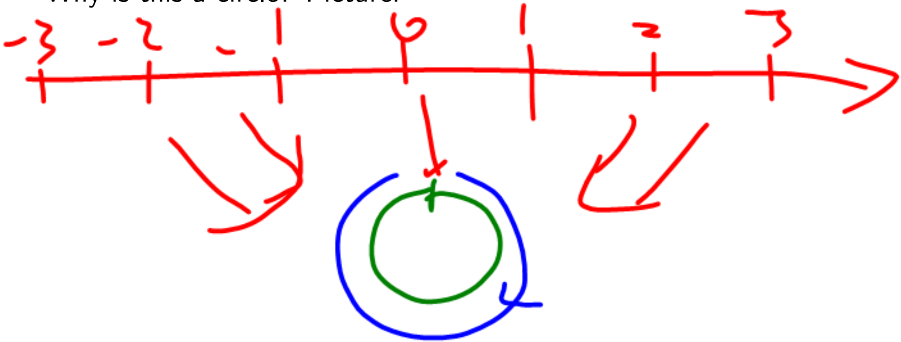
Spaces of periodic functions

Definition

To say that the domain of a function f is S^1 means:

- The domain of f is $\mathbb{R}$; and
- For all $x \in \mathbb{R}$, $f(x+1) = f(x)$, i.e., f is periodic with period 1.

Why is this a circle? Picture:



Function spaces on S^1

Continuity, limits, and derivatives defined as usual.
Integral: To say that $f : S^1 \to \mathbb{C}$ is integrable means that

$$\int_{S^1} f(x)\,dx = \int_0^1 f(x)\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\,dx$$

(any $[a, a+1]$)

exists.
Again, we have:

$$\mathcal{R}(S^1) \supset C^0(S^1) \supset C^1(S^1) \supset C^2(S^1) \supset \cdots \supset C^\infty(S^1).$$

Metrics on function spaces

One important idea we'll use a lot is the idea of putting a metric on a function space, i.e., a function that determines the distance between two functions in the space.

Definition

X a closed and bounded subset of $\mathbb{C}$ and $f, g \in C^0(X)$. We define

$$d(f,g) = \sup\{|f(x) - g(x)| \mid x \in X\}.$$

I.e., $d(f,g)$ is the worst-case scenario of the difference between $f(x)$ and $g(x)$.

Theorem

For X a closed and bounded subset of $\mathbb{C}$, $d(f,g)$ defines a metric on $C^0(X)$.

$\boxed{L^\infty \text{ metric}}$

Dot products

Dot product $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined to be

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n$$

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$.
For $v, w, x \in \mathbb{R}^n$ and $c \in \mathbb{R}$, we have

- $v \cdot w = w \cdot v$. symmetric
- $(v + w) \cdot x = v \cdot x + w \cdot x$. linear in each variable
- $(cv) \cdot w = c(v \cdot w)$.
- If $x = (x_1, \ldots, x_n)$, then $x \cdot x = x_1^2 + \cdots + x_n^2$.

  x dot x is the squared length of x

Orthogonality

Can use dot products to define not just length, but also angles.

- If $v, w \in \mathbb{R}^n$, to say that v and w are **orthogonal** means that $v \cdot w = 0$.
- To say that $\{v_1, \ldots, v_n\}$ is **orthonormal** means:

$$v_i \cdot v_j = \begin{cases} 1 & \text{if } i = j, \quad \text{normal} \\ 0 & \text{if } i \neq j. \quad \text{ortho} \end{cases}$$

An orthonormal set of size n gives "unit coordinate axes" for $\mathbb{R}^n$. Coordinates with respect to those unit coordinate axes can be conveniently computed: If $\{v_1, \ldots, v_n\}$ is an orthonormal set in $\mathbb{R}^n$ and

$$w = a_1 v_1 + \cdots + a_n v_n$$

for some $w \in \mathbb{R}^n$, then $a_i = w \cdot v_i$.

(exercise)

Summary

To study functions on S^1 (functions on $\mathbb{R}$ that are periodic with period 1):

smooth fns

- We look at a function space V like $C^0(S^1)$, $C^1(S^1)$, $C^\infty(S^1)$.
- Define a metric $d(f, g)$ on V based on something like mean squared error between f and g.
- Surprise: It turns out that the distance $d(f, g)$ is then closely related to a generalized dot product! So we can do geometry and orthogonality in V.

Trignometric polynomials

$e_n(x) = e^{2\pi i n x} = \cos(2\pi n x) + i\sin(2\pi n x)$

We finally define our central objects of study!

Definition

A **trigonometric polynomial of degree** N is $p : S^1 \to \mathbb{C}$ of the form

$$p(x) = \sum_{n=-N}^{N} c_n e_n(x)$$

$= c_{-N} e_{-N}(x) + c_{-N+1} e_{-N+1}(x) + \cdots + c_N e_N(x)$

for some **coefficients** $c_n \in \mathbb{C}$, where $e_n(x) = e^{2\pi i n x}$.

Q: Which trigonometric polynomials best approximate a given $f : S^1 \to \mathbb{C}$ *on average*?

Good approximations must have same integral properties

For $p(x) = \sum_{n=-N}^{N} c_n e_n(x)$ to approximate $f(x)$ well, should have same integral on S^1. Better yet, should have same "integral against $e_n(x)$", i.e., we want

Want $\int_0^1 p(x)\,dx = \int_0^1 f(x)\,dx$

$$\int_0^1 p(x)\overline{e_n(x)}\,dx = \int_0^1 f(x)\overline{e_n(x)}\,dx.$$

Theorem (PSOG)

For $-N \leq n \leq N$, we have

$p(x)$ Avg N

$$\int_0^1 p(x)\,\overline{e_n(x)}\,dx = c_n.$$

$p(x) = \sum_{n=-N}^{N} c_n e_n(x)$

Therefore, we guess that a trig poly that approximates f well on average will have $c_n = \int_0^1 f(x)\overline{e_n(x)}\,dx$.

The Nth Fourier polynomial of f

$e_k(x) = e^{2\pi i k x}$

Definition

Let $f : S^1 \to \mathbb{C}$ be integrable. For $n \in \mathbb{Z}$, we define

$$\boxed{\hat{f}(n) = \int_0^1 f(x)\,\overline{e_n(x)}\,dx}$$

to be the ***n*th Fourier coefficient of** f. We define the ***N*th Fourier polynomial** f_N of f to be

$$f_N(x) = \sum_{n=-N}^{N} \hat{f}(n) e_n(x).$$

In other words, $f_N(x)$ is the trigonometric polynomial of degree N whose coefficients are the Fourier coefficients $\hat{f}(n)$.

Fourier series

Definition

$f : S^1 \to \mathbb{C}$ integrable, and recall

$$\hat{f}(n) = \int_0^1 f(x)\,\overline{e_n(x)}\,dx.$$

We define the **Fourier series of** f to be:

$$f(x) \sim \lim_{N\to\infty} f_N(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e_n(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e_n(x).$$

Note that $\sim$ has no implications about convergence, pointwise or otherwise.

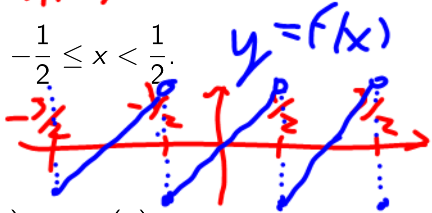
MAIN Q: When does $\sum_{n\in\mathbb{Z}} \hat{f}(n)e_n(x)$ converge to $f(x)$? (Better Q: And in what sense?)

Example

$e_0(x) = 1$ $\qquad \overline{e_n(x)} = e^{-2\pi i n x}$

Let $f : S^1 \to \mathbb{C}$ be given by

$$f(x) = x \qquad \text{for } -\frac{1}{2} \leq x < \frac{1}{2}.$$



Find the Fourier series of f.
Recall that for $n \neq 0$:

$$\int x\,\overline{e_n(x)}\,dx = -\frac{x e_{-n}(x)}{2\pi i n} - \frac{e_{-n}(x)}{(2\pi i n)^2} + C$$

When you compute Fourier coefficients, do n=0 separately.

$n=0$

$$\hat{f}(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} x\,dx = \frac{x^2}{2}\Big]_{-1/2}^{1/2}$$

$$= \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)^2\right) = 0$$

$n \neq 0$

$$\hat{f}(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} x \, \overline{e_n(x)} \, dx$$

$$= -\frac{x e_{-n}(x)}{2\pi i n} - \frac{e_{-n}(x)}{(2\pi i n)^2} \Bigg]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= -\frac{e_{-n}(\frac{1}{2})}{4\pi i n} - \frac{e_{-n}(\frac{1}{2})}{(2\pi i n)^2}$$

$$+ \left(-\frac{1}{2}\right) \frac{e_{-n}(-\frac{1}{2})}{2\pi i n} + \frac{e_{-n}(-\frac{1}{2})}{(2\pi i n)^2}$$

$$e_{-n}\left(\tfrac{1}{2}\right) = e^{2\pi i n\left(\frac{1}{2}\right)} = e^{\pi i n} = (-1)^n$$

$$e_{-n}\left(-\tfrac{1}{2}\right) = (-1)^n$$

$$= -\frac{(-1)^n}{4\pi i n} - \cancel{\frac{(-1)^n}{(2\pi i n)^2}} - \frac{(-1)^n}{4\pi i n} + \cancel{\frac{(-1)^n}{(2\pi i n)^2}}$$

$$= -\frac{(-1)^n}{2\pi i n}$$

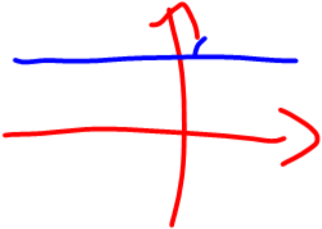
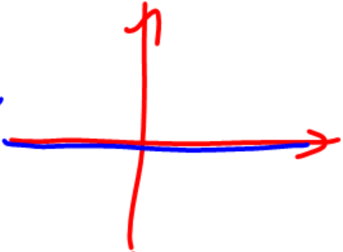
$$\hat{f}(1) = \frac{1}{2\pi i}$$

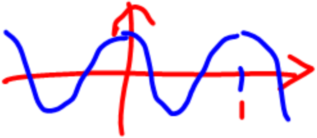
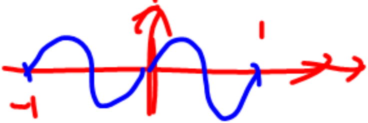
$$\hat{f}(2) = -\frac{1}{4\pi i}$$

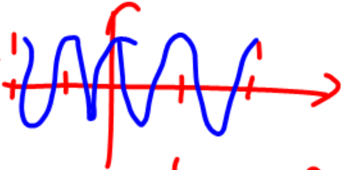
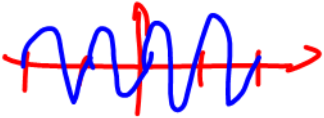
$$n \neq 0 \quad \hat{f}(n) = -\frac{(-1)^n}{2\pi i n}$$

$$\hat{f}(-1) = -\frac{1}{2\pi i}$$

$$\hat{f}(-2) = +\frac{1}{4\pi i}$$

$e_0(x) =$  $+\, i$ 

$e_1(x) =$  $+\, i$ 

$e_2(x) =$  $+\, i$ 

$$f(x) \sim \dots - \frac{1}{2\pi i} e_{-1} + 0\, e_0(x) + \frac{1}{2\pi i} e_1 - \frac{1}{4\pi i} e_2$$

Let's be less ambitious

Before we can answer:

$$\textbf{MAIN Q: } \textit{When does } \sum_{n\in\mathbb{Z}} \hat{f}(n)e_n(x) \textit{ converge to } f(x)?$$

And in what sense?

Let's tackle:

$$\textit{When does } \sum_{n\in\mathbb{Z}} \hat{f}(n)e_n(x) \textit{ converge?}$$

The surprising key:

Theorem

For $f \in C^1(S^1)$ and $n \in \mathbb{Z}$, we have that

$$\widehat{f'}(n) = (2\pi i n)\hat{f}(n).$$

Proof: PS06. (Parts!!!!)

Differentiability implies decay of coefficients

A broadly useful principle!

Theorem

For $f : S^1 \to \mathbb{C}$, we have that:

1. *If f is continuous (i.e., $f \in C^0(S^1)$), then there exists some constant $K_0 > 0$, independent of n, such that $\left|\hat{f}(n)\right| \leq K_0$ for all $n \in \mathbb{Z}$.*
2. *For any integer $r \geq 1$, if $f \in C^r(S^1)$, then there exists some constant $K_r > 0$, independent of n, such that $\left|\hat{f}(n)\right| \leq \dfrac{K_r}{|n|^r}$ for all $n \in \mathbb{Z}$, $n \neq 0$.*

Proof: