Math 131B, Wed Sep 30

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 5.1–5.3. Reading for Mon: 6.1–6.2.
- PS05 due Mon.
- Problem session Fri Oct 02, 10:00–noon on Zoom.

Review: Setting up a continuity proof (limit proofs similar)

X, Y metric spaces, $a \in X$, $f : X \to Y$, and for all $x \in X$,

$$d(f(x), f(a)) \leq \sqrt[3]{d(x, a)}.$$

Prove f is continuous at a.

Note: If we were relying on given continuity of g, h, etc., we would need a delta_1, delta_2, etc. that encodes continuity of g, h, etc.

A_1 $\epsilon > 0$

Pick $\delta(\epsilon) = \epsilon^3$

A_3 $d(x, a) < \delta(\epsilon) = \epsilon^3$

$\Rightarrow \sqrt[3]{d(x, a)} < \epsilon$

$$\Rightarrow d(f(x),f(a)) \le \sqrt[3]{d(x,a)} < \epsilon$$

$\boxed{C_3}$ $d(f(x),f(a)) < \epsilon$

$\boxed{C_2}$ If $d(x,a) < \delta(\epsilon)$ then $d(f(x),f(a))$

C_1 $\exists \delta(\epsilon) > 0$ s.t. if ... $< \epsilon$

C $\forall \epsilon > 0$

$\exists \delta(\epsilon) > 0$ st.

If $d(x,a) < \delta(\epsilon)$

then $d(f(x),f(a)) < \epsilon$

Scratch

Want $d(f(x),f(a)) < \epsilon$

$d(f(x),f(a)) \le \sqrt[3]{d(x,a)} < \epsilon$

$d(x,a) < \epsilon^3$

So $\delta(\epsilon) = \epsilon^3$

The functions $e_n(x)$

Instead of $E(z)$, $C(x)$, and $S(x)$, we can now write e^z, $\cos x$, and $\sin x$.

Definition

For $n \in \mathbb{Z}$, we define $e_n : \mathbb{R} \to \mathbb{C}$ by

$$e_n(x) = e^{2\pi i n x}.$$

Note: $\overline{e_n(x)} = \overline{e^{2\pi i n x}} = e^{-2\pi i n x} = e_{-n}(x)$.

So

$$e_n'(x) = (2\pi i n) e_n(x)$$

~~And $e_n(x)$ has period:~~

And each of the e_n is periodic with period 1.

Integration formulas

$$\overline{e_n} = e_{-n}$$

We have

$$\int \overline{e_n(x)}\, dx = -\frac{e_{-n}(x)}{2\pi i n} + C$$

$$\int x\, \overline{e_n(x)}\, dx = -\frac{x e_{-n}(x)}{2\pi i n} - \frac{e_{-n}(x)}{(2\pi i n)^2} + C$$

parts!

and so on. More importantly:

$$\underbrace{\int_0^1 e_n(x)\, \overline{e_k(x)}\, dx = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{otherwise.} \end{cases}}_{\text{PSOS}}$$

Special values of $e_n(x)$

$$|e_n(x)| = 1$$

$$e_n(k) = e^{2\pi i n k} = e_{-n}(k) = e^{-2\pi i n k} = 1$$

$$e_n\left(\frac{1}{2}\right) = e^{\pi i n} = e_{-n}\left(\frac{1}{2}\right) = e^{-\pi i n} = (-1)^n$$

$$e_n\left(\frac{1}{4}\right) = e_{-n}\left(-\frac{1}{4}\right) = e^{\pi i n/2} = i^n$$

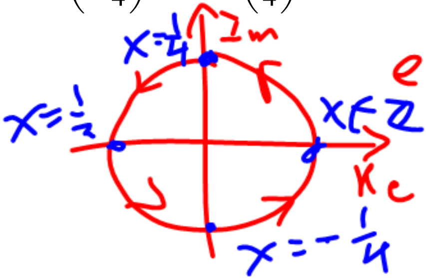
$$e_n\left(-\frac{1}{4}\right) = e_{-n}\left(\frac{1}{4}\right) = e^{-\pi i n/2} = (-i)^n$$

$e^{i\theta}$

θ rads

$e^{2\pi i x}$

x revs



$e_1(x) = e^{2\pi i x}$

Welcome to the heart of the course

In Ch. 5 & 6-8

Three big themes:

- Measuring how different two functions are
- Looking at spaces of functions instead of individual functions
- Geometry through dot products

How close are two functions?

I have two clocks: one doesn't go at all, and the other loses a minute a day: which would you prefer?

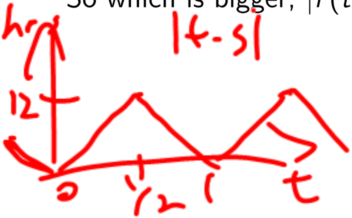
t=0 all correct

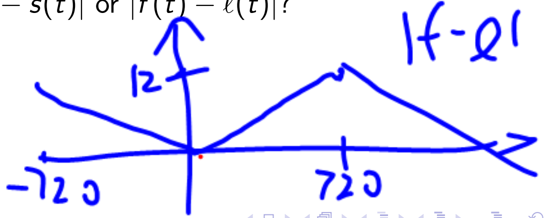
t time in days, $f(t)$ actual correct time, $s(t) = 0$ time on the stopped clock, $\ell(t)$ time on the lagging clock.
At time t, magnitudes (absolute value) of the error (in hours) of the stopped and lagging clocks are

$$|f(t) - s(t)| = |24t| \qquad \text{for } -\tfrac{1}{2} \le t \le \tfrac{1}{2},$$

$$|f(t) - \ell(t)| = \left|\frac{t}{60}\right| \qquad \text{for } -720 \le t \le 720,$$

So which is bigger, $|f(t) - s(t)|$ or $|f(t) - \ell(t)|$?





A more precise question

What do we mean by a "bigger" error?
How do we measure how close f and g are?

Which clock is closer to being correct (smaller error) **on average**?
Recall that the average value of $f(t)$ over all $t \in [a, b]$ is:

$$\frac{1}{b-a}\int_a^b f(t)\, dt.$$

So average ~~value~~ error question becomes, which is larger:

$$\frac{1}{(\frac{1}{2})-(-\frac{1}{2})}\int_{-\frac{1}{2}}^{\frac{1}{2}} |f(t)-s(t)|\ dt \qquad \text{or}$$

$$\frac{1}{720-(-720)}\int_{-720}^{720} |f(t)-\ell(t)|\ dt?$$

What happens if you take mean squared error (same, but square integrand)?

PS05

Function spaces

$(X \subseteq \mathbb{C})$

Definition

X a set. A **function space on** X is a collection V of functions, each w/domain X, such that:

1. (Nonempty) V contains the **zero function** $0(x) = 0$.
2. (Closed under addition) For $f, g \in V$, $f + g \in V$.
3. (Closed under scalar multiplication) For $f \in V$ and $c \in \mathbb{C}$, $cf \in V$.

A subset of V that is itself a function space is called a function subspace, or simply a **subspace**, of V.

In lin alg, def w/ thm of subsp.

Examples

$\mathcal{R}$ = Riemann

$\emptyset \neq X \subseteq \mathbb{C}$, every point of X is a limit point.

- If $X = [a, b]$, $\mathcal{R}(X)$ = set of all functions integrable on X.
- $C^0(X)$ = set of all continuous $f : X \to \mathbb{C}$.
- $C^r(X)$ = set of all $f : X \to \mathbb{C}$ with continuous rth derivatives.
- $C^\infty(X)$ = set of all $f : X \to \mathbb{C}$ with rth derivatives for every $r > 0$. ← calc I space!

We have: $X = [a,b]$

$$\mathcal{R}(X) \supset C^0(X) \supset C^1(X) \supset C^2(X) \supset \cdots \supset C^\infty(X).$$

$\int$-ble ← cont ← diff

$f \in C^1 \Rightarrow f'$ cont.

diff $\Rightarrow$ cont $\Rightarrow$ $\int$-ble

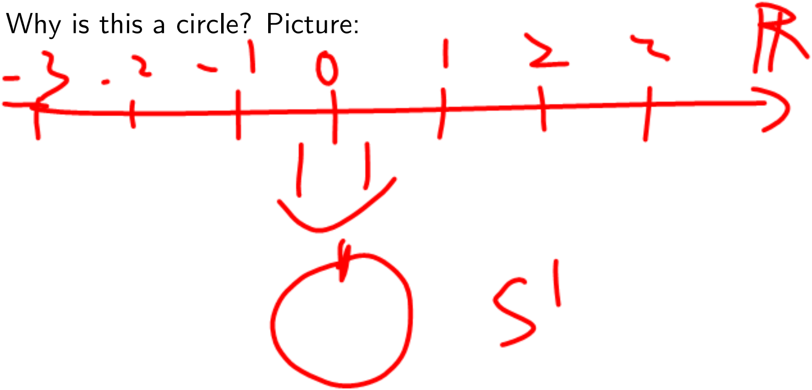
Spaces of periodic functions

Definition

To say that the domain of a function f is S^1 means: $f : \mathbb{R} \to \mathbb{C}$

- The domain of f is $\mathbb{R}$; and
- For all $x \in \mathbb{R}$, $f(x+1) = f(x)$, i.e., f is periodic with period 1.

Why is this a circle? Picture:

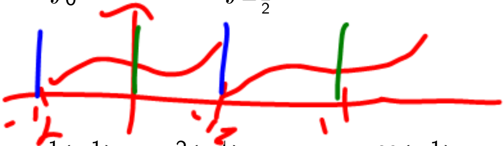


Function spaces on S^1

Continuity, limits, and derivatives defined as usual.
Integral: To say that $f : S^1 \to \mathbb{C}$ is integrable means that

$$\int_{S^1} f(x)\,dx = \int_0^1 f(x)\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\,dx$$

exists.
Again, we have:



$$\mathcal{R}(S^1) \supset C^0(S^1) \supset C^1(S^1) \supset C^2(S^1) \supset \cdots \supset C^\infty(S^1).$$

Metrics on function spaces

One important idea we'll use a lot is the idea of putting a metric on a function space, i.e., a function that determines the distance between two functions in the space.

Definition

X a closed and bounded subset of $\mathbb{C}$ and $f, g \in C^0(X)$. We define

$$d(f,g) = \sup\{|f(x) - g(x)| \mid x \in X\}.$$

I.e., $d(f,g)$ is the worst-case scenario of the difference between $f(x)$ and $g(x)$.

Theorem

For X a closed and bounded subset of $\mathbb{C}$, $d(f,g)$ defines a metric on $C^0(X)$.

Dot products

Dot product $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined to be

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1 y_1 + \cdots + x_n y_n$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{R}^n$.
For $v, w, x \in \mathbb{R}^n$ and $c \in \mathbb{R}$, we have

- $v \cdot w = w \cdot v$.
- $(v + w) \cdot x = v \cdot x + w \cdot x$.
- $(cv) \cdot w = c(v \cdot w)$.
- If $x = (x_1, \dots, x_n)$, then $x \cdot x = x_1^2 + \cdots + x_n^2$.

Orthogonality

Can use dot products to define not just length, but also angles.

- If $v, w \in \mathbb{R}^n$, to say that v and w are **orthogonal** means that $v \cdot w = 0$.
- To say that $\{v_1, \dots, v_n\}$ is **orthonormal** means:

$$v_i \cdot v_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

An orthonormal set of size n gives "unit coordinate axes" for $\mathbb{R}^n$. Coordinates with respect to those unit coordinate axes can be conveniently computed: If $\{v_1, \dots, v_n\}$ is an orthonormal set in $\mathbb{R}^n$ and

$$w = a_1 v_1 + \cdots + a_n v_n \tag{1}$$

for some $w \in \mathbb{R}^n$, then $a_i = w \cdot v_i$.

Summary

To study functions on S^1 (functions on $\mathbb{R}$ that are periodic with period 1):

- We look at a function space V like $C^0(S^1)$, $C^1(S^1)$, $C^\infty(S^1)$.
- Define a metric $d(f, g)$ on V based on something like mean squared error between f and g.
- The distance $d(f, g)$ will then be closely related to a generalized dot product! So we can do geometry and orthogonality in V.