Math 131B, Mon Sep 28

And we've run out of 1st round of music, so I'll be asking for more music requests....

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 4.5–4.6. Reading for Wed: 5.1–5.3.
- PS04 due tonight. PS05 outline due Wed night.
- Problem session Fri ~~Sep 25~~, 10:00–noon on Zoom.

Oct 02

How to prove that $\cancel{f_n} \to f$ uniformly

$\sum g_n$

Our main technique:

Theorem (Weierstrass M-test)

$g_n : X \to \mathbb{C}$ be sequence of functions, and M_n a sequence of nonnegative real numbers such that $\sum M_n$ converges and

$$|g_n(z)| \leq M_n$$

M_n dominates $g_n(z)$

for all $z \in X$. Then $\sum_{n=0}^{\infty} g_n(z)$ converges absolutely and uniformly to some $f : X \to \mathbb{C}$.

M_n for **majorant**, something bigger than $g_n(z)$ for all z. Basically the comparison test for series of functions.

Example

Let

$$f(x) = \sum_{n=0}^{\infty} \overbrace{x^n \sin(nx)}^{g_n(x)}.$$

Prove that the series converges uniformly on $\left[0, \frac{1}{2}\right]$.

Pf Use M-test. Let $M_n = \left(\frac{1}{2}\right)^n$

$$|x^n \sin(nx)| = |x^n|\,|\overset{\le 1}{\sin(nx)}|$$

What is the worst-case scenario for how big this gets?

$$\le |x^n|$$

Since x is in [0,1/2]

$$\le \left(\frac{1}{2}\right)^n = M_n$$

But $\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ convs, b/c geom series w/ $|r|<1$. So by M-test,

$$\sum_{n=0}^{\infty} x^n \sin(nx)$$ convs abs & unif.

You'll do this on 4.4.1 in PS04 and many other times this semester.....

Power series

4.4

The fundamentals from Analysis I (upgraded to Analysis II)
* Limits of sequences
* Continuity, differentiability, integration, FTC
* Infinite series, series of functions, uniform convergence

4.4, 4.5, 4.6 are all about recovering calc I and calc II, extended to complex numbers in places.
Goal is for you to be able to do the rest yourself!

Definition

A **power series** is a (complex-valued) series of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where the $a_n \in \mathbb{C}$ are the **coefficients** of the power series, and we interpret z^0 as the constant function 1.

The radius of convergence

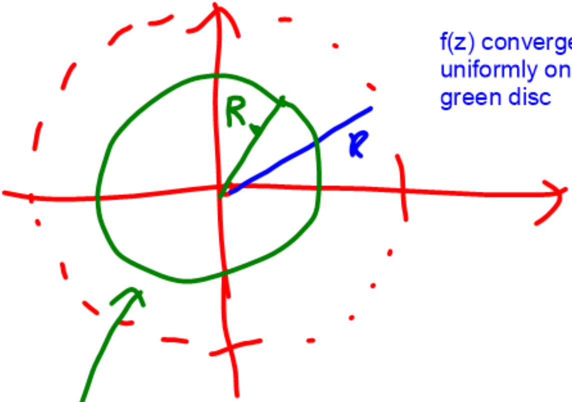
Theorem

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series such that $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists, and let $R = \frac{1}{\rho}$, where we define $R = \infty$ when $\rho = 0$. Then:

1. *For any R_0 such that $0 \le R_0 < R$, the power series $f(z)$ converges uniformly on the closed disc $\overline{\mathcal{N}_{R_0}(0)}$.*
2. *It follows that $f(z)$ converges pointwise (but not necessarily uniformly) on the open disc $\mathcal{N}_R(0)$.*
3. *Let $b_n = na_n$. Then $\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \rho$ as well.*
4. *It follows that $f(z)$ is differentiable on $\mathcal{N}_R(0)$, and that*

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} z^k$$

term by term differentiation



f(z) converges abs & uniformly on closed green disc

Part 4 also says that we can do term by term differentiation inside radius of convergence.

Exponential functions 4.5

Now you can go figure out calculus yourself!

Definition

For $z \in \mathbb{C}$, we define

Once we establish the familiar properties of exponential fn, we'll use notation e^z instead of E(z).

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) z^n.$$

Theorem

Design: Use power series here, and then only use E'(z)=E(z), E(0)=1.

The power series $E(z)$ has radius of convergence $R = \infty$. Furthermore, $E(0) = 1$, $\overline{E(z)} = E(\bar{z})$, and for all $z \in \mathbb{C}$, $E'(z) = E(z)$.

$\frac{d}{dz}(e^z) = e^z$

Theorem

$e^z \neq 0$

For any $z \in \mathbb{C}$, $E(z) \neq 0$.

Theorem

$e^{z+w} = e^z e^w$

For $z, w \in \mathbb{C}$, we have that $E(z + w) = E(z)E(w)$.

$\checkmark E' = E \quad E(0) = 1$ $\checkmark$

Theorem

Enuf to show that E(z)E(-z)=1.

For any $z \in \mathbb{C}$, $E(z) \neq 0$.

Proof: Let $f(z) = E(z)E(-z)$.

$$f'(z) = \frac{d}{dz}(E(z))E(-z) + E(z)\frac{d}{dz}(E(-z))$$

$$= E'(z)E(-z) + E(z)\left[E'(-z)\cdot(-1)\right]$$

$$= E(z)E(-z) - E(z)E(-z) = 0$$

So $f(z) = E(z)E(-z) = \text{const.}$

$f(0) = E(0)E(0) = 1$. So $E(z)E(-z) = 1$

Trig functions

Definition

Define $C : \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ by

$$E(ix) = C(x) + iS(x)$$

for all $x \in \mathbb{R}$.

$e^{ix} = \cos x + i \sin x$ is our def'n of cos, sin

Theorem

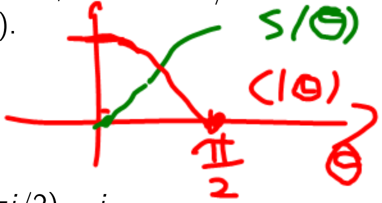
1. $C(0) = 1$ *and* $S(0) = 0$.
2. $C(-x) = C(x)$ *and* $S(-x) = -S(x)$.
3. $|E(ix)| = 1$ *and* $C(x)^2 + S(x)^2 = 1$.
4. $C'(x) = -S(x)$ *and* $S'(x) = C(x)$.

All you need about C, S.

Periodicity of trig functions

Definition

We define $\pi = 2 \inf V$, or in other words, we define $\pi/2$ to be the infimum of all positive zeros of $C(x)$.



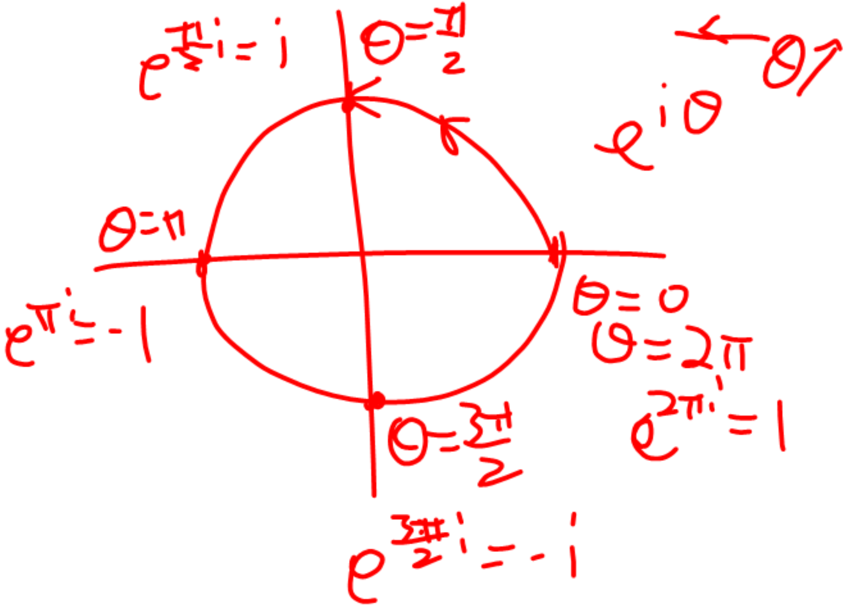
Theorem

We have that:

1. $C(\pi/2) = 0$.
2. $S(\pi/2) = 1$, *and therefore,* $E(\pi i/2) = i$.
3. $E(2\pi i) = 1$. $\quad e^{2\pi i} = 1$
4. *For any* $x \in \mathbb{R}$, $E(i(x + 2\pi)) = E(ix)$.

$\boxed{e^{ix} \text{ has pd } 2\pi.}$

Picture:



$e^{\frac{\pi}{2}i} = i$
$\theta = \frac{\pi}{2}$
$e^{i\theta}$
θ
$\theta = \pi$
$e^{\pi i} = -1$
$\theta = 0$
$\theta = 2\pi$
$e^{2\pi i} = 1$
$\theta = \frac{3\pi}{2}$
$e^{\frac{3\pi}{2}i} = -i$

The functions $e_n(x)$

$\boxed{4.6}$

Instead of $E(z)$, $C(x)$, and $S(x)$, we can now write e^z, $\cos x$, and $\sin x$.

$\frac{d}{dx}(e^{2x}) = 2e^{2x}$

Definition

For $n \in \mathbb{Z}$, we define $e_n : \mathbb{R} \to \mathbb{C}$ by

$$e_n(x) = e^{2\pi i n x}.$$

Note: $\overline{e_n(x)} = \overline{e^{2\pi i n x}} = e^{-2\pi i n x} = e_{-n}(x)$.

So

$$e_n'(x) = (2\pi i n) e_n(x)$$

$$e_n''(x) = -4\pi^2 n^2 e_n(x)$$

And $e_n(x)$ has period:

e^{ix} has period 2π

$e_1(x) = e^{2\pi i x}$ has period 1

$e_n(x) = e^{2\pi i n x}$ has pd $\frac{1}{|n|}$

$(n \neq 0)$

e^{ix} has period 2pi, i.e., to go around unit circle once, x goes from 0 to 2pi.

The function e^{i(2 pi x)} goes around unit circle once as 2 pi x goes from 0 to 2 pi, i.e., once as x goes from 0 to 1.

Integration formulas

We have

$$\int \overline{e_n(x)}\,dx = -\frac{e_{-n}(x)}{2\pi in} + C$$

$$\int x\,\overline{e_n(x)}\,dx = -\frac{xe_{-n}(x)}{2\pi in} - \frac{e_{-n}(x)}{(2\pi in)^2} + C$$

and so on. More importantly:

$$\int_0^1 e_n(x)\,\overline{e_k(x)}\,dx = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{otherwise.} \end{cases}$$

Special values of $e_n(x)$

$$|e_n(x)| = 1$$

$$e_n(k) = e^{2\pi ink} = e_{-n}(k) = e^{-2\pi ink} = 1$$

$$e_n\left(\frac{1}{2}\right) = e^{\pi in} = e_{-n}\left(\frac{1}{2}\right) = e^{-\pi in} = (-1)^n$$

$$e_n\left(\frac{1}{4}\right) = e_{-n}\left(-\frac{1}{4}\right) = e^{\pi in/2} = i^n$$

$$e_n\left(-\frac{1}{4}\right) = e_{-n}\left(\frac{1}{4}\right) = e^{-\pi in/2} = (-i)^n$$

unit circle!