Math 131B, Wed Sep 23

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 4.3–4.4. Reading for Mon: 4.5–4.6.
- Outline for PS04 due tonight; completed version due Mon Sep 28.
- Problem session Fri Sep 25, 10:00–noon on Zoom.

The six NO's

$\forall z \in X, f_n(z) \to f(z)$

Suppose f_n converges pointwise to f on a domain X.

QB If the f_n are all bounded on X, must f be bounded on X?

QC If the f_n are all continuous on X, must f be continuous on X?

QD1 If the f_n are all differentiable on X, must f be differentiable on X?

QD2 If the f_n and f are all differentiable on X, must it be the case that f_n' converges pointwise to f' on X?

QI1 If the f_n are all integrable on X, must f be integrable on X?

QI2 If the f_n and f are all integrable on $[a, b]$, must it be the case that $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$?

Back to Maple

To the Maple worksheet. . . .

How can we fix these pointwise problems?

Defn: $f_n : X \to \mathbb{C}$, $f : X \to \mathbb{C}$ functions.
To say that f_n converges **uniformly** to f on X means:

$\forall \epsilon > 0$, $\exists N(\epsilon)$ s.t.:
$\forall z \in X$
If $n > N(\epsilon)$
then $|f_n(z) - f(z)| < \epsilon$

As usual, uniform convergence of a series $\sum_{n=0}^{\infty} g_n(z)$ is defined in terms of the uniform convergence of its sequence of partial sums

$$f_N(z) = \sum_{n=0}^{N} g_n(z).$$

Pointwise vs. uniform convergence

Think of N as rate of convergence as n -> infinity.

$f_n(z) \to f(z)$ pointwise on X:

$\forall z \in X \quad \forall \epsilon > 0$

$\exists N(\epsilon, z)$

If $n > N(\epsilon, z)$

then $|f_n(z) - f(z)| < \epsilon$.

For pointwise convergence, rate of convergence can be different at different points in domain.

$f_n(z) \to f(z)$ uniformly on X:

$\forall \epsilon > 0$

$\exists N(\epsilon)$ s.t.

$\forall z \in X$

here, N cannot depend on z!!!

If $n > N(\epsilon)$

then $|f_n(z) - f(z)| < \epsilon$.

For uniform convergence, rate of convergence is same at different points in domain. (Or actually, there is a worst case that works everywhere.)

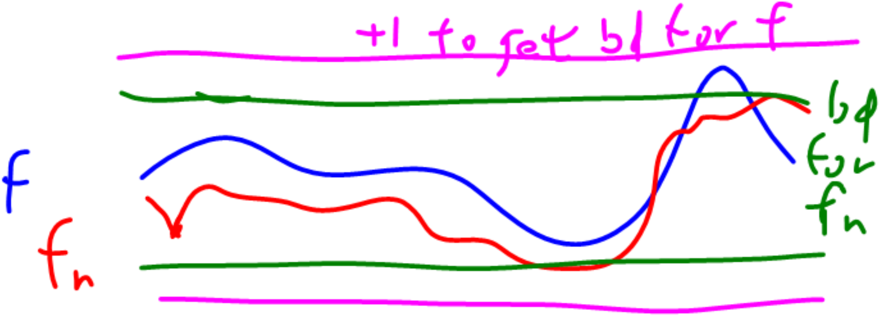
First Uniform YES: QB

Theorem

$f_n : X \to \mathbb{C}$ a sequence of functions, each bounded on X, such that f_n converges uniformly on X to some $f : X \to \mathbb{C}$. Then f is bounded on X.

Proof: Picture:

For some fixed n, f is within 1 of f_n everywhere in the domain.



Pf By defn of unif conv, $\epsilon = 1$, $\exists N(1)$ s.t. if $n > N(1)$, $z \in X$, then (A)

$$|f_n(z) - f(z)| < 1.$$

Fix some $n > N(1)$.

f_n is bd, so $\exists M$ s.t. $\forall z \in X$,

$$|f_n(z)| < M.$$

$\boxed{A}$ $z \in X$

$$|f(z)| \le |f(z) - f_n(z)| + |f_n(z)|$$

by (*)

$$< 1 + M$$

C $|f(z)| < M + 1$

C $\forall z \in X, \ |f(z)| < M + 1$

so f bd.

Uniform YES: QC

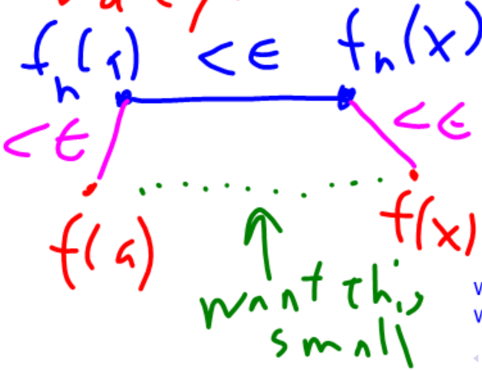
PS04

Note: To bound the distance travelled along a path by a sum of distances of other paths, use triangle inequality.

Theorem

$f_n : X \to \mathbb{C}$ a sequence of functions, each continuous on X, such that f_n converges uniformly on X to some $f : X \to \mathbb{C}$. Then f is continuous on X.

Picture: $\forall a \in X$



We know that f_n(x) is close to f_n(a) when x is close to a

We know that f(a) close to f_n(a), and f(x) SAME close to f_n(x), by unif conv.

want f(x) to be close to f(a) when x is close to a

Uniform YES: QI1

TeXT

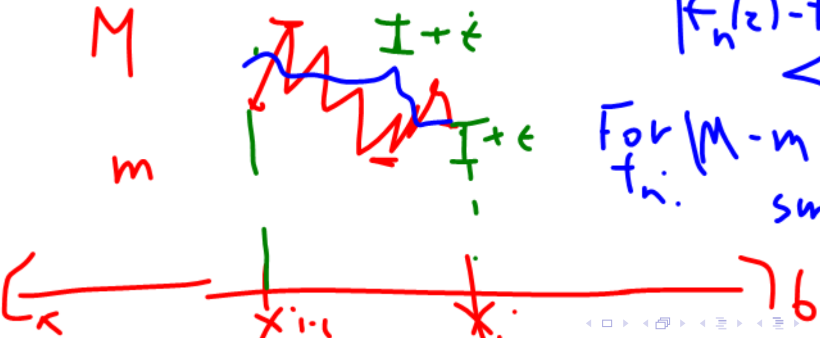
Theorem

Let $f_n : [a, b] \to \mathbb{C}$ be a sequence of functions, each integrable on $[a, b]$, such that f_n converges uniformly on $[a, b]$ to some $f : [a, b] \to \mathbb{C}$. Then f is integrable on $[a, b]$.

$(M-m)\Delta x_i$

Picture: (real case)

To prove f integrable, need to bound "weighted maximum wiggle" on each subinterval.



$|f_n(z) - f(z)| < \epsilon$

For f_n: $|M-m|\Delta x_i$ small

On each subint, $|M-m|\Delta x_i$

incr by $\le 2\epsilon\Delta x_i$.

So total added to $U-L$ is

$$\sum 2\epsilon\Delta x_i = 2\epsilon\sum\Delta x_i$$

$$= 2\epsilon(b-a).$$

Since $2(b-a)$ const, still small.

Uniform YES: QI2

P504

Theorem

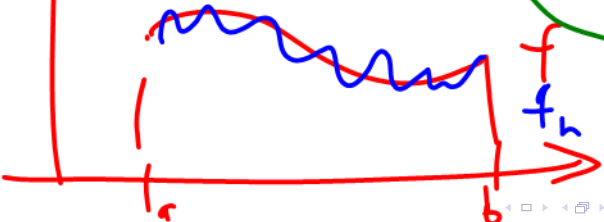
Let $f_n : [a, b] \to \mathbb{C}$ be a sequence of functions, each integrable on $[a, b]$, such that f_n converges uniformly on $[a, b]$ to some $f : [a, b] \to \mathbb{C}$. Then

$$\int_a^b f(x)\,dx = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right)\,dx = \lim_{n\to\infty}\int_a^b f_n(x)\,dx.$$

For any epsilon, we need to find an n s.t. the integral of f_n is within epsilon of integral of f.

Picture: (real case)

By unif conv, can choose f_n s.t. f_n is within epsilon of f for all x in [a,b].



So $\int_a^b f_n$ differs from $\int_a^b f$ by at most $\epsilon(b-a)$.

Purple area in the middle is at most epsilon*(b-a)

Still no: QD1, QD2

Back to Maple....

Uniform f_n': QD1 and QD2 yes

Theorem

Suppose don't need f_n to converge unif

- $f_n : X \to \mathbb{C}$ *seq of differentiable functions that converges pointwise to* $f : X \to \mathbb{C}$.
- *Each* f_n' *continuous and* f_n' *converges uniformly to some* $g : X \to \mathbb{C}$. DO need their derivatives to conv unif

Then f *is differentiable on* X *and* $f'(z) = g(z)$, *i.e.,*

$$\frac{d}{dz}\left(\lim_{n\to\infty} f_n(z)\right) = \lim_{n\to\infty}\left(\frac{d}{dz} f_n(z)\right).$$

Proof in textbook.

How to prove that $f_n \to f$ uniformly

Our main technique:

Theorem (Weierstrass M-test)

$g_n : X \to \mathbb{C}$ be sequence of functions, and M_n a sequence of nonnegative real numbers such that $\sum M_n$ converges and

$$|g_n(z)| \leq M_n$$

for all $z \in X$. Then $\sum_{n=0}^{\infty} g_n(z)$ converges absolutely and uniformly to some $f : X \to \mathbb{C}$.

M_n for **majorant**, something bigger than $g_n(z)$ for all z. Basically the comparison test for series of functions.

Case: Power series

Definition

A **power series** is a (complex-valued) series of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where the $a_n \in \mathbb{C}$ are the **coefficients** of the power series, and we interpret z^0 as the constant function 1.

The radius of convergence

Theorem

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series such that* $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ *exists, and let* $R = \frac{1}{\rho}$*, where we define* $R = \infty$ *when* $\rho = 0$*. Then:*

1. *For any* R_0 *such that* $0 \le R_0 < R$*, the power series* $f(z)$ *converges uniformly on the closed disc* $\overline{\mathcal{N}_{R_0}(0)}$*.*
2. *It follows that* $f(z)$ *converges pointwise (but not necessarily uniformly) on the open disc* $\mathcal{N}_R(0)$*.*
3. *Let* $b_n = na_n$*. Then* $\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \rho$ *as well.*
4. *It follows that* $f(z)$ *is differentiable on* $\mathcal{N}_R(0)$*, and that*

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} z^k$$