Math 131B, Mon Sep 14

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for Wed Sep 23: 4.3–4.4.
- Exam review Fri Sep 18, 10:00–noon on Zoom. 128A 10am, 131B 11am; session will be recorded.
- **Exam 1 Mon Sep 21**, on 2.1–2.5, 3.1–3.4, i.e., PS01–03.
- Outline for PS04 due Wed Sep 23.

Exam procedure for Mon Sep 21

1. Please have a clear workspace ready where you can write.
2. Please have some kind of camera ready. First position the camera so I can see your face, and later so I can see your workspace.
3. Please have the Gradescope assignment page "Exam 1" open and ready to go.
4. Exam will be handed out via chat, or by email if necessary.

Questions?

Ch. 4: Infinite series in $\mathbb{C}$

Defn: a_n seq in $\mathbb{C}$. Series $\sum_{n=k}^{\infty} a_n$ defined as:

- Define sequence of partial sums s_N by setting $s_k = a_k$ and, for $N \geq k$, setting $s_{N+1} = s_N + a_{N+1}$. In other words:

$$\begin{aligned}
s_k &= a_k \\
s_{k+1} &= s_k + a_{k+1} = a_k + a_{k+1} \\
s_{k+2} &= s_{k+1} + a_{k+2} = a_k + a_{k+1} + a_{k+2} \\
s_{k+3} &= s_{k+2} + a_{k+3} = a_k + a_{k+1} + a_{k+2} + a_{k+3} \\
&\vdots \\
s_{N+1} &= s_N + a_{N+1} = a_k + a_{k+1} + \cdots + a_{N+1} \\
&\vdots
\end{aligned}$$

Ch. 4: Infinite series in $\mathbb{C}$

- To say that $\sum_{n=k}^{\infty} a_n$ converges means that the sequence of partial sums s_N converges, in which case we define

I.e., sum of an infinite series is limit of its partial sums.

$$\sum_{n=k}^{\infty} a_n = \lim_{N\to\infty} s_N.$$

Can also do a two-sided version of the above to define the sum

$$\sum_{n\in\mathbb{Z}} a_n = \sum_{n=1}^{\infty} a_n + \sum_{n=0}^{-\infty} a_n,$$

which converges iff both sums on RHS converge. See book for details.

Two-sided series are important for us because Fourier series are two-sided series!

$$\sum_{n\in\mathbb{Z}} a_n = \cdots + a_{-2} + a_{-1} + a_0 + a_1 + a_2 + \cdots$$

Cauchy criterion for series

This is a restatement of seq of partial sums being Cauchy

Cauchy completeness of $\mathbb{C}$ implies:

Corollary (Cauchy Criterion for Series)

The series $\sum a_n$ *converges if and only if for every* $\epsilon > 0$, *there exists some* $N(\epsilon)$ *such that if* $m, k \in \mathbb{Z}$ *and* $m, k > N(\epsilon)$, *then*

$$\left| \sum_{n=k}^{m} a_n \right| < \epsilon.$$

$$= |S_m - S_{k-1}|$$

b/c $S_m = a_0 + a_1 + \cdots + a_{k-1} + a_k + \cdots + a_m$

$S_{k-1} = a_0 + a_1 + \cdots + a_{k-1}$

Comparison and absolute convergence

Σb_n known, real
Σa_n unknown

Corollary (Comparison Test)

a_n, b_n sequences, $b_n \geq 0$.

1. *If $\sum b_n$ converges and $|a_n| \leq b_n$ for all n (or sufficiently large n), then $\sum a_n$ converges.* If big series converges, so does little one.
2. *If $\sum b_n$ diverges, $a_n \geq 0$, and $b_n \leq a_n$ for all n (or sufficently large n), then $\sum a_n$ diverges.* If little series diverges, so does big one.

Pf Cauchy.

Corollary

If $\sum |a_n|$ converges, then so does $\sum a_n$. □

Use abs conv to simplify analysis of series w/complex terms.

Definition

To say that $\sum a_n$ **converges absolutely** means that $\sum |a_n|$ converges (and therefore, so does $\sum a_n$).

Special cases of series

To use Comparison, we need series whose convergence we understand, so we can Compare new series to known series. Same for absolute convergence.

Example (Geometric series)

$\sum r^n$ converges if and only if $|r| < 1$, in which case

Nth partial

$= \frac{1-\cancel{r^{n+1}}}{1-r}$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}. \tag{1}$$

Example (p-series)

For a real number $p > 0$, we call $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ a p-**series**. Fact:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges if } p > 1, \\ \text{diverges if } 0 < p \leq 1. \end{cases}$$

And there are two-sided versions of both of those.

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{(i)^n}{n^{3/2}}$ conv/div?

<u>Ans</u> $\displaystyle\left|\frac{(i)^n}{n^{3/2}}\right| = \frac{1}{n^{3/2}}$

$|i|=1$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is p-series w/ $p=\frac{3}{2}>1$, so conv.

So $\displaystyle\sum_{n=1}^{\infty} \frac{i^n}{n^{3/2}}$ conv absolutely.

NB: Poss that $\sum |a_n|$ div, $\sum a_n$ conv.

(This is called conditional convergence, as opposed to absolute convergence.)

Define $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

$\zeta(2)$ $\zeta(4)$: PS09 or 10

$\zeta(3)$ = Ph.D. $\zeta(5)$ $\zeta(7)$ = tenure-track prof

All $\zeta(\text{odd})$ — Fields medal

We haven't yet defined 2^i, but we will after Ch. 4.

Absolute convergence implies that zeta(s) converges for Re(s)>1.

But we can actually extend zeta(s) to all complex s. That's Riemann zeta function of the Riemann hypothesis, which is the most famous unsolved problem in math....

Ratio test

By comparison with geometric series, we have:

Theorem (Ratio Test)

Suppose a_n is a sequence such that $a_n \neq 0$ and $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = r$.

Then:

1. *If $r < 1$, then $\sum a_n$ converges absolutely.*
2. *If $r > 1$ then $\sum a_n$ diverges.*

If $r=1$, ☹.

Σa_n like Σr^n.

4.2: Sequences and series of functions

E.g. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \sin(2\pi n x)$

Definition

Let X be a nonempty subset of $\mathbb{C}$, let $f_n : X \to \mathbb{C}$ be a sequence of functions, and let $f : X \to \mathbb{C}$ be a function. To say that the sequence f_n **converges pointwise** to f means that for any fixed $z \in X$, $\lim_{n\to\infty} f_n(z) = f(z)$. Pointwise convergence then defines **series of functions** in the same way that convergence of sequences is used to define ordinary series: namely, to say that $\displaystyle\sum_{n=0}^{\infty} g_n(z)$ converges pointwise on X means that for each fixed $z \in X$, $\displaystyle\sum_{n=0}^{\infty} g_n(z)$ is a convergent series, or in other words, the partial sums

$$f_N(z) = \sum_{n=0}^{N} g_n(z)$$ converge pointwise to some $f : X \to \mathbb{C}$.

Interesting to us because Fourier series are series of functions.

I.e. By defn $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin(2\pi n x)$

$= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^2} \sin(2\pi n x)$

So if we want to understand infinite series of functions, we first have to understand the convergence of sequences of functions.

Things you might hope are true for sequences of functions

For $z \in X$, $\lim_{n\to\infty} f_n(z) = f(z)$

Suppose f_n converges pointwise to f on a domain X.

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NOs

- QB If the f_n are all bounded on X, must f be bounded on X?
- QC If the f_n are all continuous on X, must f be continuous on X?
- QD1 If the f_n are all differentiable on X, must f be differentiable on X?
- QD2 If the f_n and f are all differentiable on X, must it be the case that f_n' converges pointwise to f' on X?
- QI1 If the f_n are all integrable on X, must f be integrable on X?
- QI2 If the f_n and f are all integrable on $[a, b]$, must it be the case that $\displaystyle\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$?

On to Maple

$$= \int_a^b \left(\lim_{n\to\infty} f_n(x)\right) dx$$

To the Maple worksheet!