Math 131B, Wed Dec 02

Colloquium 3pm today: Stephanie Salomone, "What I Believe" Especially of interest to future teachers!

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- ► Reading for today: 12.4. THE END
- Otline for PS11 due tonight; full version due Mon Dec 07.
- Problem session, Fri Dec 04, 10:00am–noon on Zoom. V510

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FINAL EXAM, MON DEC 14, 9:45am-noon.

Recap Laplace transform FT for C Definition Schurch $t \neq z$ $\mathcal{S}(\mathbb{R})$ is the space of all $t \in \mathbb{R} \to \mathbb{C}$ such that for all $k \geq 0$, the *k*th derivative $f^{(k)}(x)$ of f exists for all $x \in \mathbb{R}$ and is rapidly decaying.

Definition

For $f \in \mathcal{S}(\mathbb{R})$, define the **Fourier transform** of *f* to be the function $\widehat{f}: \mathbb{R} \to \mathbb{C}$ given by

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \gamma x} dx$$

for any $\gamma \in \mathbb{R}$. $\gamma = f \gamma \in \mathcal{S}(\mathbb{R})$, integral definitely converges.

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Last time
$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - t)g(t) dt$$
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• Convolution $f * g : \mathbb{R} \to \mathbb{C}$ defined by
 $(f * g)(x) = \int_{-\infty}^{\infty} f(x - t)g(t) dt$.
• Dirac kernel $K_t : \mathbb{R} \to \mathbb{R}$ $(t \in \mathbb{R}, t > 0)$:
• For all $t > 0$ and all $x \in \mathbb{R}$, $K_t(x) \ge 0$;
• For all $t > 0$, $\int_{-\infty}^{\infty} K_t(x) dx = 1$; and
• For fixed $\eta > 0$, we have $\lim_{t \to 0^+} \int_{|x| \ge \eta} K_t(x) dx = 0$.
• Thm: $\lim_{t \to 0^+} (f * K_t)(x) = f(x)$.
• Example of a Dirac kernel: Gauss kernel
 $G_t(x) = \frac{1}{t} \exp\left(\frac{-\pi x^2}{t^2}\right)$.

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Properties of Fourier transform

	Function (in x)	Fourier transform (in γ)
U	f(x+a)	$e^{2\pi i a \gamma} \hat{f}(\gamma)$
6	$e^{2\pi iax}f(x)$	$\widehat{f}(\gamma-a)$
(3)	f(-x)	$\widehat{f}(-\gamma)$

In operator notation:

$$U(f) = \hat{f} \qquad (s_{-1}(f))(x) = f(-x) (\tau_a(f))(x) = f(x+a) \qquad (\mu_a(f))(x) = e^{2\pi i a x} f(x)$$

The above table says:

$$\begin{array}{c} \textcircled{0} \\ U(\tau_{a}(f)) = \mu_{a}(U(f)) \\ \textcircled{0} \\ U(s_{-1}(f)) = s_{-1}(U(f)) \end{array} \begin{array}{c} \textcircled{0} \\ U(\mu_{a}(f)) = \tau_{-a}(U(f)) \\ \textcircled{0} \\ U(s_{-1}(f)) = s_{-1}(U(f)) \end{array}$$

I.e., $U\tau_a = \mu_a U$, $U\mu_a = \tau_{-a}U$, $Us_{-1} = s_{-1}U$.

Recall: How to write Fourier transform with freqency variable same as the variable you started with (time variable)

flx) e S(R) -> f(x)= \$ f(y) e->#iny dy Then U: S(R) -> S(R) $U(f) = \hat{f}$

"Pass the hat" and the Gauss kernel

Theorem Passtheffat If $f, g \in S(\mathbb{R})$, then $\int_{-\infty}^{\infty} \hat{f}(x)g(x) dx = \int_{-\infty}^{\infty} f(x)\hat{g}(x) dx$. PSIO Theorem The Fourier transform of $f(x) = e^{-\pi x^2}$ is $\hat{f}(\gamma) = e^{-\pi \gamma^2}$. In other words, f is its own Fourier transform, or U(f) = f. More generally, for t > 0, let $G_t(x) = \frac{1}{t} \exp\left(\frac{-\pi x^2}{t^2}\right)$ be the Gauss kernel. Then $U(U(G_t)) = \hat{\hat{G}}_t = G_t$ $\hat{G}_t(\gamma) = e^{-\pi t^2 \gamma^2}$

An ugly lemma

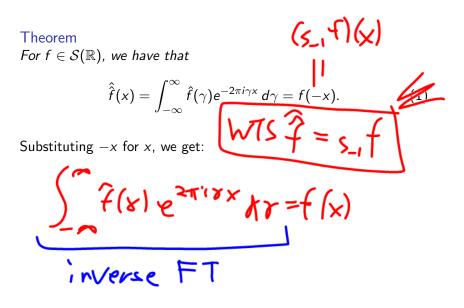
Lemma For $f \in S(\mathbb{R})$ and constant $x \in \mathbb{R}$, let $h_x(y) = f(-x - y)$. Then $\hat{f}(x - y) = \hat{h}_x(y)$, where the Fourier transform is calculated in the variable y. For clarity, \hat{h}_x means $U(U(h_x))$.

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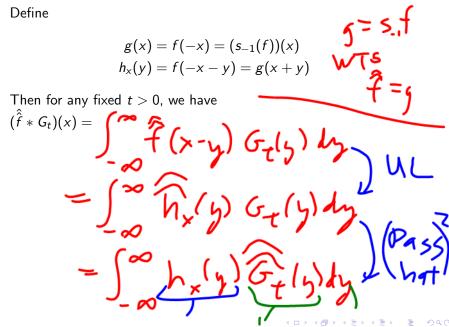
Proof: First get $h_x(y)$ by applying operators to f(y): $h_x(y) = f(-x-y) = f(-(y+x))$ $=(s_{1}(f))(y_{1}+x)=T_{x}(s_{1}(f))(y_{1})$ Then get $\hat{f}(x-y)$ by applying operators to f(y): f(x-y) = (U(U(f)))(x-y) = (U(U(f)))(-(y-x))= (s_, (y (y (p))) (y-x)=(t-~s_, h up))

Then previous operator facts give: Much, C-x5-,UUF $= (N_{M_{\tau_x}S_{-1}}(f))$ - Mrzus,(+) = 2-x UUS-1(f) fn w/for mula the $-2-xS_1UN(f),$ 7(x-y). ・ロト ・四ト ・ヨト ・ヨト ・ヨー

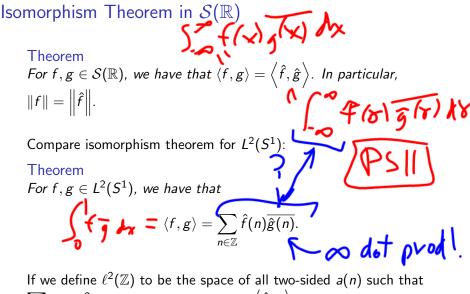
Inversion Theorem in $\mathcal{S}(\mathbb{R})$



Proof of inversion



 $= \int_{-\infty}^{\infty} g(x+y) G_{\tau}(y) dy \int_{-\infty}^{\infty} G_{t} = G_{t}$ $= \int_{\infty}^{\infty} g(x-u) G_{\tau}(u) (-du) \qquad M^{-1} M_{\tau}$ $= \int_{\infty}^{\infty} g(x-u) G_{\tau}(u) \ du$ $= \int_{-\infty}^{\infty} g(x-y) \, \mathcal{G}_{\overline{z}}(y) \, dy$ So: $\widehat{f} = \lim_{t \to 0^+} \widehat{f} * G_t = \lim_{t \to 0^+} g * G_t = g$



 $\sum_{n \in \mathbb{Z}} |a(n)|^2 < \infty, \text{ then above RHS is } \left\langle \hat{f}, \hat{g} \right\rangle.$

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