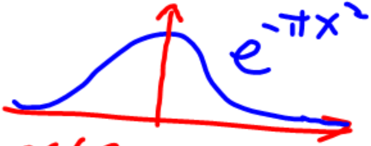
Math 131B, Wed Nov 18

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 12.2. Reading for Mon Nov 30: 12.3.
- Outline for PS10 due Fri Nov 20; PS10 due ~~Fri Nov 30.~~ Mon Nov 30
- Problem session/exam review, Fri Nov 20, **9:00–11:00am** on Zoom. 131B segment starts at **9:00am**.
- **EXAM 3, MON NOV 23.**

$\boxed{\text{PS07-09}}$

Recap



Definition

Schwartz space

$\mathcal{S}(\mathbb{R})$ is the space of all $f : \mathbb{R} \to \mathbb{C}$ such that for all $k \geq 0$, the kth derivative $f^{(k)}(x)$ of f exists for all $x \in \mathbb{R}$ and is rapidly decaying.

$f \in C^\infty$

$\to 0$ faster $\frac{1}{|x|^m}$

Definition

For $f \in \mathcal{S}(\mathbb{R})$, define the **Fourier transform** of f to be the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ given by

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \gamma x}\, dx$$

Compare

$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x}\, dx$

for any $\gamma \in \mathbb{R}$.

Note that because we now assume $f \in \mathcal{S}(\mathbb{R})$, integral definitely converges.

The plan (12.2)

think: t like $\frac{1}{n}$

Almost the same plan as the proof of the inversion theorem:

1. Convolutions
2. Dirac kernels K_t
3. Prove $\lim_{t \to 0^+} (f * K_t)(x) = f(x)$.

   (series: $\lim_{n \to \infty} f * K_n = f$)
4. Specific example of a Dirac kernel (Gauss kernel)

Convolution

Definition $(S(\mathbb{R}))$

For $f, g \in L^2(\mathbb{R})$, the **convolution** $f * g : \mathbb{R} \to \mathbb{C}$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - t)g(t)\,dt.$$

Compare the version for $f, g \in L^2(S^1)$:

$$(f * g)(x) = \int_0^1 f(x - t)g(t)\,dt.$$

sum is x; var of $\int$ is t

Properties of convolution

O. (pf omitted)

Theorem

If $f, g \in C^0(\mathbb{R})$ are rapidly decaying, then $\underbrace{f * g \text{ is rapidly decaying}}$. Moreover, suppose $f, g, h \in \mathcal{S}(\mathbb{R})$. Then:

PS10 1. $(f * g)(x) = (g * f)(x)$. Comm
2. $((f * g) * h)(x) = (f * (g * h))(x)$. Assoc.

PS10 3. $\dfrac{d}{dx}((f * g)(x)) = \left(\dfrac{df}{dx} * g\right)(x)$. Smoothing

4. $f * g \in \mathcal{S}(\mathbb{R})$.

Proof of last property, assuming previous ones:

$\boxed{A}$ $f, g \in S(\mathbb{R})$

Prop 3 $\Rightarrow$ $(f*g)' = (f'*g)$ exists on $\mathbb{R}$.

B/c $f \in S(\mathbb{R})$, $f' \in S(\mathbb{R})$. (b/c $f' \in C^\infty$ and derivs r.d.)

$\Rightarrow$ $(f' * g)$ rapid decay. (Prop 0)

Prop 3 $\Rightarrow$ $(f*g)'' = (f'' * g)$ exists

B/c $f' \in S(\mathbb{R})$, $f'' \in S(\mathbb{R})$ And so on...

$\Rightarrow$ $(f'' * g)$ rapid decay (induction)

$\boxed{C}$ For $k \geq 0$, $x \in \mathbb{R}$, $(f*g)^{(k)}(x)$ exists and $(f*g)^{(k)}(x)$ rapid decay.

$\therefore$ $f * g \in S(\mathbb{R})$

Dirac kernel ($\mathbb{R}$ version)

Definition

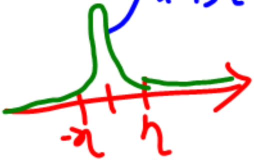
A **Dirac kernel on** $\mathbb{R}$ is a family of continuous $K_t : \mathbb{R} \to \mathbb{R}$ ($t \in \mathbb{R}$, $t > 0$) integrable on $\mathbb{R}$ s.t.:

K_t is prob dist.

1. For all $t > 0$ and all $x \in \mathbb{R}$, $K_t(x) \geq 0$; nonneg
2. For all $t > 0$, $\displaystyle\int_{-\infty}^{\infty} K_t(x)\,dx = 1$; and area 1
3. For any fixed $\eta > 0$, we have

(Concentrated at 0)

$$\lim_{t \to 0^+} \int_{|x| \geq \eta} K_t(x)\,dx = 0.$$



I.e., for $\eta > 0$, $\epsilon > 0$, $\exists\ \delta(\eta, \epsilon) > 0$ s.t. for $0 < t < \delta(\eta, \epsilon)$,

$$1 - \epsilon < \int_{-\eta}^{\eta} K_t(x)\,dx \leq 1.$$

See Maple.

Key property of Dirac kernel s

on S':

$$\lim_{n\to\infty} (f * K_n)(x) = f(x)$$

unif on S'

Theorem

If $\{K_t\}$ is a Dirac kernel, and $f \in \mathcal{S}(\mathbb{R})$, then

$$\lim_{t\to 0^+} (f * K_t)(x) = f(x)$$

uniformly on $\mathbb{R}$ (i.e., with convergence independent of $x \in \mathbb{R}$).

Proof uses two lemmas.

Lemmas for key property of Dirac kernel

Lemma 1

For any $\epsilon_1 > 0$, there exists some $\eta_1(\epsilon_1) > 0$ such that for $0 < \eta < \eta_1(\epsilon_1)$, any $x \in \mathbb{R}$, and any $t > 0$, we have

$$\int_{-\eta}^{\eta} \overbrace{|f(x-y) - f(x)|}^{\text{make this small}} |K_t(y)| \, dy < \epsilon_1 .$$

Lemma 2

For any fixed $\eta > 0$ and $\epsilon_2 > 0$, there exists some $\delta_2(\eta, \epsilon_2)$ such that for $0 < t < \delta_2(\eta, \epsilon_2)$ and any $x \in \mathbb{R}$, we have

$$\int_{|y| \geq \eta} |f(x-y) - f(x)| |K_t(y)| \, dy < \epsilon_2 .$$

$$\int_{-\infty}^{-\eta} + \int_{\eta}^{\infty}$$

We prove the first lemma.

④ $\epsilon_1 > 0$ Fact (4.7) $f \in S(\mathbb{R}) \to f$ unif cont.

So $\forall \epsilon_0 > 0, \exists \delta_0(\epsilon_0)$ s.t. if $|u-v| < \delta_0(\epsilon_0)$, then
$|f(u) - f(v)| < \epsilon_0$.

Let $\eta_1(\epsilon_1) = \delta_0\left(\frac{\epsilon_1}{2}\right)$.

Ⓐ $0 < \eta < \eta_1(\epsilon_1), t > 0$

Then for $|y| \leq \eta < \eta_1(\epsilon_1)$,

$|(x-y) - x| = |y| < \eta_1(\epsilon_1) = \delta_0\left(\frac{\epsilon_1}{2}\right)$

so $|f(x-y) - f(x)| < \frac{\epsilon_1}{2}$.

So: $\int_{-\eta}^{\eta} |f(x-y)-f(x)| K_t(y)\,dy$

$\leq \int_{-\eta}^{\eta} \left(\frac{\epsilon_1}{2}\right) K_t(y)\,dy$

$= \left(\frac{\epsilon_1}{2}\right) \boxed{\int_{-\eta}^{\eta} K_t(y)\,dy} \leq \frac{\epsilon_1}{2} < \epsilon_1$

$\int_{-\eta}^{\eta} |f(x-y)-f(x)| K_t(y)\,dy < \epsilon_1$

Proof of key property of Dirac kernels

Theorem

If $\{K_t\}$ is a Dirac kernel, and $f \in \mathcal{S}(\mathbb{R})$, then

$$\lim_{t \to 0^+} (f * K_t)(x) = f(x)$$

uniformly on $\mathbb{R}$ (i.e., with convergence independent of $x \in \mathbb{R}$).

Sketch of proof:

$$|(f * K_t)(x) - f(x)|$$

$$\leq \cdots \leq \int_{-\infty}^{\infty} |f(x-y) - f(x)| |K_t(y)| \, dy$$

$\forall \epsilon > 0$ Let $\epsilon_1 = \frac{\epsilon}{2}$, $\epsilon_2 = \frac{\epsilon}{2}$.

$\epsilon_1 \Rightarrow \eta_1(\epsilon_1)$; pick $0 < \eta < \eta_1(\epsilon_1)$

η & $\epsilon_2 \Rightarrow \delta_2(\epsilon_2, \eta)$

Let $\delta(\epsilon) = \delta_2(\epsilon_2, \eta)$

(A) $0 < t < \delta(\epsilon)$

$\lim_{t \to 0}$

Lemma 1 & Lem 2

$\Downarrow$

(C) $|(f * K_t)(x) - f(x)| < \epsilon$.

The Gauss kernel

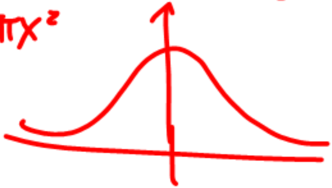
$\exp(y) = e^y$

Example

The **Gauss kernel** $\{G_t\}$ is

$$G_t(x) = \frac{1}{t}\exp\left(\frac{-\pi x^2}{t^2}\right) = \frac{1}{t} e^{-\pi x^2/t^2}.$$

For example, $G_1(x) = e^{-\pi x^2}$



See Maple.

Gauss kernel works

Theorem

The Gauss kernel G_t is a Dirac kernel.

Recall (PS10): $\displaystyle\int_{-\infty}^{\infty} G_1(x) = \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1.$

Other properties of G_t follow from substitution (!!); see PS10.