Math 131B, Mon Nov 16

- Use a laptop or desktop with a large screen so you can read these words clearly.
- In general, please turn off your camera and mute yourself.
- Exception: When we do groupwork, please turn both your camera and mic on. (Groupwork will not be recorded.)
- Please always have the chat window open to ask questions.
- Reading for today: 4.7, 4.8, 12.1. Reading for Wed: 12.2.
- PS09 due today. Outline for PS10 due Fri Nov 20.
- Problem session/exam review, Fri Nov 20, **9:00–11:00am** on Zoom.
- **EXAM 3, MON NOV 23.**

PS7-9, Ch7-8

Extra Derivative Lemma

Lemma

If $g \in L^2(S^1)$, then the two-sided series

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left(\frac{1}{2\pi n}\right)\hat{g}(n)$$

converges absolutely (as a series of complex numbers).

Proof:

HSACT: $\sum c_n e_n$ convs to $f \in L^2(S^1)$ $\iff$ $\sum |c_n|^2$ convs in $\mathbb{R}$

L^2 inv: $\sum \hat{g}(n) e_n(x)$ conv to g.

HS ACT $\Rightarrow \sum |\hat{g}(n)|^2$ convs

HS ACT $\Longrightarrow$ $\sum_{n\in\mathbb{Z}} |\hat{g}(n)| c_n(x)$ convs to some $h \in L^2(S^1)$

$\sum_{n\neq 0} \left|\frac{1}{2\pi n}\right|^2$ convs by p-series $(p=2>1)$

$\overset{HSACT}{\Longrightarrow}$ $\sum_{n\neq 0} \left|\frac{1}{2\pi n}\right| e_n(x)$ convs to some $f \in L^2(S^1)$.

So, since $\langle f,h\rangle$ is finite, and

$\langle f,h\rangle$ $\left(\frac{\text{Isom}}{\text{Thm}}\right)$

$$= \sum_{n \neq 0} \left| \frac{1}{2\pi n} \right| |\hat{g}(n)|$$

Isom Thm for Hilbert Spaces: B/c {e_n} are an orthonormal basis, we can compute inner products as dot products of Fourier coefficients.

Convs. Ie.

$$\sum_{n \neq 0} \left(\frac{1}{2\pi n}\right)(\hat{g}(n))$$ convs. abs.

This is called the XDL b/c we were able to prove that C^2 functions had absolutely convergent Fourier series with our "bare hands" (Ch. 6).

The XDL allows us to prove the same for C^1 functions:

C^1 uniform convergence

Bare hands: If $f \in C^2(S^1)$, F.S. of f conv to something.

Theorem

Suppose $f \in C^1(S^1)$. Then the Fourier series of f converges absolutely and uniformly to f.

you

Proof: We first show that the Fourier series of f converges absolutely and uniformly to some $g \in C^0(S^1)$. (This is on PS09.)

Therefore, for all $n \in \mathbb{Z}$, $\hat{g}(n) = \hat{f}(n)$. However, then implies that $f = g$ a.e., and since both f and g are continuous, measure 0 stuff implies that $f = g$ **everywhere**.

Sect 7.4: If f, g continuous, and equal almost everywhere, then f=g everywhere.

Summary

L^2 **inversion theorem:** If $f \in L^2(S^1)$, then the Fourier series of f converges to f **in the L^2 metric.** In other words, we can recover f completely (in L^2) from $\hat{f}(n)$. $f(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$

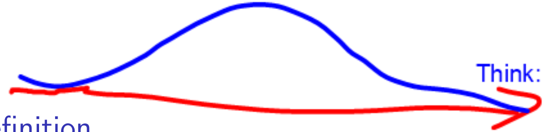
However, f is only recovered "up to a set of measure zero".

f' cont on S^1

C^1 **inversion theorem:** If $f \in C^1(S^1)$, then the Fourier series of f converges **absolutely and uniformly** to f.

4.7: The Schwartz space

= nicest possible functions whose domain is R



Think: f decays exponentially as x -> +/- infinity

Definition

To say that a continuous function $f : \mathbb{R} \to \mathbb{C}$ is **rapidly decaying** means that one of the following equivalent conditions holds:

1. For any $n \geq 0$, $x^n f(x)$ is bounded on $\mathbb{R}$.
2. For any $n \geq 0$, $\lim_{x \to \pm\infty} x^n f(x) = 0$.

I.e., as $x \to \pm\infty$, $f(x)$ goes to 0 faster than any function $1/x^n$.

$\boxed{\text{Defn}}$

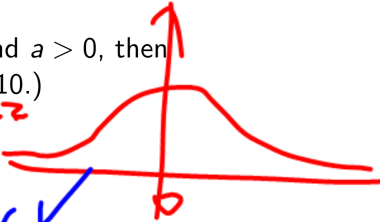
$\mathcal{S}(\mathbb{R})$ is the space of all $f : \mathbb{R} \to \mathbb{C}$ such that for all $k \geq 0$, the kth derivative $f^{(k)}(x)$ of f exists for all $x \in \mathbb{R}$ and is rapidly decaying.

$(\text{So } f \in C^\infty(\mathbb{R}))$

(Non)examples in $\mathcal{S}(\mathbb{R})$

THE example: If $p(x)$ is a polynomial and $a > 0$, then $f(x) = p(x)e^{-ax^2+bx}$ is in $\mathcal{S}(\mathbb{R})$. (See PS10.)

$e^{-\pi x^2}$



Non-examples:

$g(x) = e^{-|x|}$

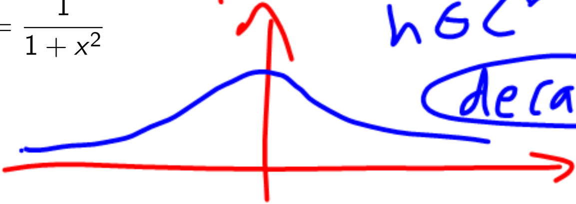
decay ✓

$g \notin C^1(\mathbb{R})$

$h(x) = \dfrac{1}{1+x^2}$

$h \in C^\infty(\mathbb{R})$ ✓

decay ✗



Integration by parts on $\mathbb{R}$

4.8

Much of calculus can be extended to (improper) integrals on all of $\mathbb{R}$, i.e., $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$, when that converges. (E.g., when $f \in \mathcal{S}(\mathbb{R})$.)
For example:

Theorem (Integration by Parts on $\mathbb{R}$)

Suppose that $f, g : \mathbb{R} \to \mathbb{C}$ are differentiable, f' and g' are continuous on $\mathbb{R}$, $f(x)g'(x)$ is integrable on $\mathbb{R}$, and both $\displaystyle\lim_{a\to-\infty} f(a)g(a)$ and $\displaystyle\lim_{b\to\infty} f(b)g(b)$ exist. Then $g(x)f'(x)$ is integrable on $\mathbb{R}$, and

$$\int_{-\infty}^{\infty} f(x)g'(x)\,dx = \underbrace{f(x)g(x)\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} g(x)f'(x)\,dx.$$

Proof omitted for time.

If $f, g \in S(\mathbb{R})$, this $= 0$.

Differentiating under the integral sign on $\mathbb{R}$

Theorem

Suppose $f(x, y) = h(x, y)k(y)$, where $|h(x, y)| \leq C$ for all $x, y \in \mathbb{R}$ and $k(y) \in \mathcal{S}(\mathbb{R})$, and suppose $\frac{\partial f}{\partial x}$ is continuous on $[a, b] \times \mathbb{R}$ (as a function of two variables). Let

$$F(x) = \int_{-\infty}^{\infty} f(x, y)\, dy.$$

Then for all $x \in [a, b]$,

$$F'(x) = \frac{d}{dx} \int_{-\infty}^{\infty} f(x, y)\, dy = \int_{-\infty}^{\infty} \frac{\partial f}{\partial x}\, dy \cancel{= g'(x)}.$$

Proof omitted for time.

Fubini's theorem on $\mathbb{R}$

$\iint$ on $\mathbb{R}^2$:

Theorem

Suppose $f, g \in \mathcal{S}(\mathbb{R})$, and $G \in C^1(\mathbb{R}^2)$ is such that G, $\dfrac{\partial G}{\partial x}$, and $\dfrac{\partial G}{\partial y}$ are all bounded. Suppose also that either $F(x,y) = G(x,y)f(x)g(y)$ or $F(x,y) = G(x,y)f(x-y)g(y)$. Then both sides of

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} F(x,y)\,dy\right)\,dx = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} F(x,y)\,dx\right)\,dy$$

converge and are equal.

Again, proofs omitted for time.

12.1: The Fourier Transform

$$\int_{\mathbb{R}} |f(x)|^2 < \infty$$

"Definition"

For $f \in L^2(\mathbb{R})$, define the **Fourier transform** of f to be the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ given by

$\hat{f}$

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} \underbrace{f(x)e^{-2\pi i \gamma x}\,dx}$$

for any $\gamma \in \mathbb{R}$.

Integrate in x, treating gamma as constant; result is a fn of gamma.

Compare

$$\hat{f}(n) = \int_0^1 f(x)e^{-2\pi i n x}\,dx$$

for $f : S^1 \to \mathbb{C}$ and $n \in \mathbb{Z}$. Just as n represents a discrete "frequency" for $f \in L^2(S^1)$, γ represents a continuous "frequency" for $f \in L^2(\mathbb{R})$.

Questions about the Fourier Transform

1. Recall that to say that $f \in L^p(\mathbb{R})$ means that

$$\int_{-\infty}^{\infty} |f(x)|^p \, dx$$

is finite. (Think $p = 1$ or 2.)
Well, if $f \in L^2(\mathbb{R})$, not obvious that

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \gamma x} \, dx$$

is finite for all $\gamma \in \mathbb{R}$, i.e., not clear that the Fourier transform $\hat{f}$ exists!

Note: If $\lim_{x\to\infty} f(x) = 0$ then $|f(x)|^2 < |f(x)|$ as $x \to \infty$

2. If $\hat{f}$ does actually exist, to what extent can we recover f from $\hat{f}$?

Goal (that we won't quite reach)

Theorem (Inversion Theorem for the Fourier Transform)
If $f \in L^2(\mathbb{R})$ and $\hat{f}(\gamma)$ is defined by

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \gamma x}\, dx,$$

then f can be recovered from $\hat{f}(\gamma)$ by the inverse Fourier transform

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\gamma) e^{2\pi i x \gamma}\, d\gamma.$$

$\hat{f}(\gamma)$

We'll only consider the case where $f \in \mathcal{S}(\mathbb{R})$, where we can do everything very concretely; the full L^2 theorem requires more abstraction.

Comparing Fourier series

Compare Inversion Theorem for Fourier Series: If $f \in L^2(S^1)$ and

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x},$$

then

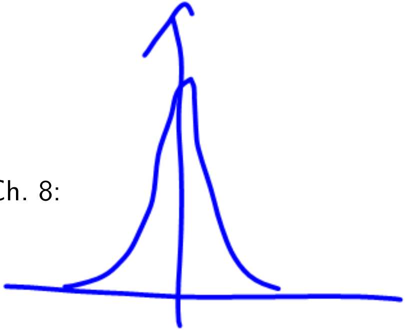
$$f(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$$

in the L^2 metric.

Helpful to keep in mind:

	Function variable	Transform variable
Fourier series	$x \in S^1$	$n \in \mathbb{Z}$
Fourier transform	$x \in \mathbb{R}$	$\gamma \in \mathbb{R}$

The plan



Use same kinds of tools as in Ch. 8:

- Convolution
- Dirac kernel G_t
- Limit of $f * G_t$ is f

Then we show that inversion theorem works for G_t, and leverage that to show that inversion theorem works in general.