1. (15 points) Let $\mathcal{H}$ be a Hilbert space, and let $\{u_n \mid n \in \mathbb{N}\}$ be a set of nonzero vectors in $\mathcal{H}$.

(a) Define what it means for $\{u_n \mid n \in \mathbb{N}\}$ to be an orthogonal set.

(b) Define what it means for $\{u_n \mid n \in \mathbb{N}\}$ to be an orthogonal basis.

(a) $\langle u_i, u_j \rangle = 0$ for all $i \neq j$

(b) $\{u_n \mid n \in \mathbb{N}\}$ orthog set and:

$$\forall f \in \mathcal{H},\ f = \sum_{n=1}^{\infty} \hat{f}(n)\, u_n = \sum_{n=1}^{\infty} \frac{\langle f, u_n \rangle}{\langle u_n, u_n \rangle} u_n$$

2. (15 points)

(a) For f in the Schwartz space $\mathcal{S}(\mathbb{R})$, define the Fourier transform $\hat{f}(\xi)$ of f.

(b) State the Fourier inversion theorem in the case of $f \in \mathcal{S}(\mathbb{R})$. (In other words, what do you need to do to $\hat{f}$ to recover f?)

(a) $\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \gamma x}\, dx$ FT

(b) $f(x) = \int_{-\infty}^{\infty} \hat{f}(\gamma) e^{2\pi i \gamma x}\, d\gamma$ InvFT

Alt: $\hat{\hat{f}}(x) = f(-x)$

10. (15 points) **PROOF QUESTION.** Suppose $f, g \in L^2(S^1)$. Prove that

$$\left| \underbrace{\int_0^1 f(x)\overline{g(x)}\, dx}_{=} \right|^2 \leq \left(\sum_{n \in \mathbb{Z}} \left| \hat{f}(n) \right|^2 \right) \left(\int_0^1 |g(x)|^2 \, dx \right).$$

LHS: $|\langle f, g \rangle|^2$ by defn

RHS: $\int_0^1 |g(x)|^2 dx = \| g \|^2$ by defn

$\sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 = \|f\|^2$ Parseval

Cauchy-Schwarz:

$|\langle f, g \rangle| \leq \|f\| \|g\|$

So $|\langle f, g \rangle|^2 \leq \|f\|^2 \|g\|^2$

13. (15 points) **PROOF QUESTION.** Suppose $f \in$ ~~$L^2(S^1)$~~ $S(\mathbb{R})$, and let $g(x) = f(-x)$. Prove that for $n \in \mathbb{Z}$, $\hat{g}(n) = \hat{f}(-n)$. (Remember that $\hat{f}$ and $\hat{g}$ here refer to Fourier series coefficients, and not the Fourier transform on $\mathbb{R}$.)

FT version! WTS $\hat{g}(\gamma) = \hat{f}(-\gamma)$

$$\hat{g}(\gamma) = \int_{-\infty}^{\infty} f(-x) e^{-2\pi i \gamma x}\, dx$$

$x \to \infty$ $u \to -\infty$ $u = -x$

$x \to -\infty$ $u \to +\infty$ $du = -dx$

$$= \int_{+\infty}^{-\infty} f(u) e^{2\pi i \gamma u} (-du)$$

cancel

$$= \int_{-\infty}^{\infty} f(u) e^{-2\pi i (-\gamma) u}\, du$$

$$= \hat{f}(-\gamma).$$

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WTS $g(x) = e^{-\pi x^2} \Rightarrow \hat{g}(\gamma) = e^{-\pi \gamma^2}$

$G_t(x) = \frac{1}{t} e^{-\pi \left(\frac{x}{t}\right)^2} = \frac{1}{t} \exp\left(\frac{-\pi x^2}{t^2}\right)$

WTS $\hat{G}_t = G_t$

(c) WTSolve $F'(\gamma) = -2\pi\gamma F(\gamma)$

$\hookrightarrow$ Solve $\frac{dy}{d\gamma} = -2\pi\gamma y$

$\frac{1}{y} dy = -2\pi\gamma \, d\gamma$

$\int$ both, + C
Solve for C
w/ F(0).

f	$\hat{f}$
$e^{-\pi x^2}$	$e^{-\pi \gamma^2}$
$\frac{1}{t} e^{-\pi\left(\frac{x}{t}\right)^2}$	$\frac{1}{t}$ (table from p.271)
$f(bx)$	$\frac{1}{b}\hat{f}\left(\frac{\gamma}{b}\right)$

d, e