Math 131B, Fall 2019 **Name:** ____________________

Final exam

This test consists of 14 questions on 12 pages, totalling 200 points. You are not allowed to use books, notes, or calculators. Unless otherwise stated, you may take as given anything which has been proven in class, in the homework, or in the reading. You are also given the information in several tables on this page and the next.

Problem	Points	Score
1	14	
2	14	
3	14	
4	13	
5	13	
6	13	
7	13	
8	13	
9	13	
10	16	
11	16	
12	16	
13	16	
14	16	
Total	200	

You may freely use the following formulas:

$$\int \overline{e_n(x)}\,dx = -\frac{e_{-n}(x)}{2\pi i n} + C$$

$$\int x\,\overline{e_n(x)}\,dx = -\frac{x e_{-n}(x)}{2\pi i n} - \frac{e_{-n}(x)}{(2\pi i n)^2} + C$$

$$\int x^2\,\overline{e_n(x)}\,dx = -\frac{x^2 e_{-n}(x)}{2\pi i n} - \frac{2x e_{-n}(x)}{(2\pi i n)^2} - \frac{2e_{-n}(x)}{(2\pi i n)^3} + C$$

$$\int_0^1 e_n(x)\,\overline{e_k(x)}\,dx = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

$$e_n(k) = e_{-n}(k) = 1 \qquad e_n\left(\frac{1}{2}\right) = e_{-n}\left(\frac{1}{2}\right) = (-1)^n$$

$$e_n\left(\frac{1}{4}\right) = e_{-n}\left(-\frac{1}{4}\right) = i^n \qquad e_n\left(-\frac{1}{4}\right) = e_{-n}\left(\frac{1}{4}\right) = (-i)^n$$

1. (14 points) Let f be in $L^2(S^1)$, and N be a positive integer.

(a) Define $f_N(x)$, the Nth Fourier polynomial of f.

(b) Define what it means for $p(x)$ to be a trigonometric polynomial of degree N.

(c) State the *Best Approximation Theorem*. (I.e., what is the most notable property of the Nth Fourier polynomial of f?)

(a) $f_N(x) = \sum_{n=-N}^{N} \hat{f}(n) e_n(x)$

(b) $p(x) = \sum_{n=-N}^{N} c_n e_n(x) \quad c_n \in \mathbb{C}$

(c) $\|f - f_N\| \leq \|f - p\|$ for any trig poly of deg N. $\widehat{p}$

2. (14 points) Suppose $f, g \in C^0(S^1)$.

(a) Define the convolution $(f * g)(x)$.

(b) What is the most notable property of the Fourier coefficients of $f*g$? State the formula precisely.

sum is x

(a) $(f*g)(x) = \int_0^1 f(x-t)\, g(t)\, dt$

FT $\int_{-\infty}^{\infty}$

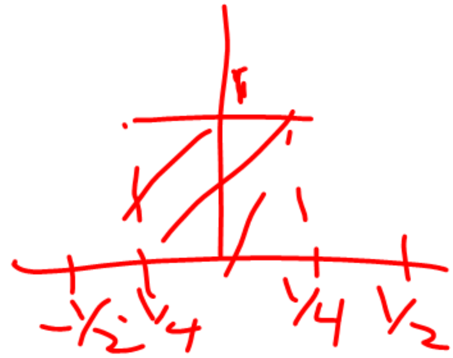
(b) $\widehat{f*g}(n) = \hat{f}(n)\hat{g}(n)$

$\widehat{f*g}(\gamma) = \hat{f}(\gamma)\hat{g}(\gamma)$

3. (14 points) Calculate the Fourier coefficients $\hat{f}(n)$ of the function $f : S^1 \to \mathbf{C}$ given for $-\frac{1}{2} \leq x \leq \frac{1}{2}$ by

$$f(x) = \begin{cases} 1 & \text{if } -\frac{1}{4} \leq x \leq \frac{1}{4}, \\ 0 & \text{otherwise.} \end{cases}$$

Show all your work, and do not simplify your final answer.

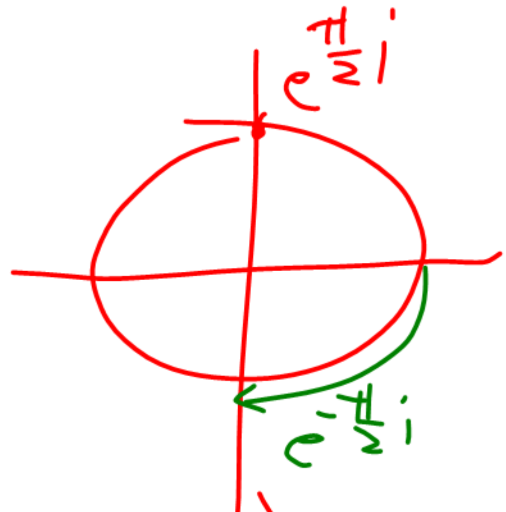


$$\hat{f}(0) = \int_{-1/4}^{1/4} 1\,dx = \frac{1}{2}$$

$n \neq 0$

$$\hat{f}(n) = \int_{-1/2}^{1/2} f(x)\,\overline{e_n(x)}\,dx = \langle f, e_n \rangle$$

$$= \int_{-1/4}^{1/4} e^{-2\pi i n x}\,dx$$



$$= \frac{e^{-2\pi i n x}}{-2\pi i n}\Bigg]_{-1/4}^{1/4}$$

$$= \frac{1}{-2\pi i n}\left(e_{-n}\left(\tfrac{1}{4}\right) - e_{-n}\left(-\tfrac{1}{4}\right)\right)$$

$$= \frac{1}{-2\pi i n}\left((-i)^n - (+i)^n\right)$$

$n = 1$: $e^{-2\pi i \left(\frac{1}{4}\right)}$

For questions 4–9, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

4. (13 points) **TRUE/FALSE.** It is possible that there exists a sequence a_n in $\mathbf{R}$ and a continuous function $f : \mathbf{R} \to \mathbf{C}$ such that $\lim_{n\to\infty} a_n = 5$, $f(5) = 13$, and $\lim_{n\to\infty} f(a_n) = 7$.

FALSE f cont

$\Rightarrow \lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$

$7 = f(5) = 13$

Contra

(Ch.2)

5. (13 points) **TRUE/FALSE.** If $f : [1, 5] \to \mathbf{C}$ is a Riemann integrable function, then it must be the case that f is continuous.

FALSE

(Ch. 3)

This is piecewise cont, so $\int$, but not cont.

(Diff $\Rightarrow$ cont $\Rightarrow$ $\int$-ble but not reverse.)

8. (13 points) **TRUE/FALSE.** Suppose $f \in L^2(S^1)$, and let f_N be the Nth Fourier polynomial of f. Then it must be the case that $\lim_{N\to\infty} \|f - f_N\| = 0$.

L^2 norm

True

Inv Thm $f \in L^2(S^1) \Rightarrow f_N \to f$ in L^2

9. (13 points) **TRUE/FALSE.** Let f_n be a sequence of continuous functions on $[0,1]$, and suppose that $f : [0,1] \to \mathbf{C}$ is a function such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [0,1]$. Then it must be the case that f is continuous.

False

$$f_n(x) = x^n$$

$$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

$f_n \to f$ ptwise, f_n cont, f not cont.

6 NOs

10. (16 points) **PROOF QUESTION.** Suppose $f \in C^3(S^1)$. Prove that

$$\sum_{n \in \mathbf{Z}} (2\pi n)\hat{f}(n)$$

converges absolutely. (Suggestion: You may find the Extra Derivative Lemma to be helpful, though it is not necessary for this problem.)

$|\hat{f}(n)| \leq \frac{K}{n^3}$

B/C $f \in C^3(S^1)$, $\exists K$ s.t. $|\hat{f}(n)| \leq \frac{K}{|n|^3}$ for $n \neq 0$.

So, for $n \neq 0$,

$$|(2\pi n)\hat{f}(n)| = |2\pi n|\,|\hat{f}(n)|$$

$$\leq |2\pi n| \frac{K}{|n|^3} = \frac{2\pi K}{|n|^2}$$

$\sum_{n \neq 0} \frac{1}{|n|^2}$ convs by p-series $(p = 2 > 1)$

So $\sum_{n \in \mathbb{Z}} 2\pi n \hat{f}(n)$ convs abs by comparison.

Ch. 4

11. (16 points) **PROOF QUESTION.** For $k = 0$ and $k = 1$, define $f_k : S^1 \to \mathbf{C}$ by

$$f_k(x) = \sum_{n\neq 0} \left(\frac{(2\pi i n)^k}{n^3}\right) e_n(x).$$

f_0, f_1

(a) Prove that if either $k = 0$ or $k = 1$, then $f_k(x)$ converges absolutely and uniformly on S^1.

M test!

(b) Prove that $f_0(x)$ is differentiable and $f_0'(x) = f_1(x)$. Be precise about the hypotheses you need to make term-by-term differentiation work.

(a) $$\left|\frac{(2\pi i n)^k}{n^3} e_n(x)\right| = \frac{|2\pi i n|^k}{|n|^3}\underbrace{|e_n(x)|}_{=1}$$

$n \neq 0$

$$= \frac{2\pi |n|^k}{|n|^3} \le \underbrace{\frac{2\pi}{|n|^2}}_{} = M_n$$

$\sum_{n\neq 0} M_n = \sum_{n \neq 0} \frac{2\pi}{|n|^2}$ convs by p-series ($p = 2 > 1$)

So $\sum_{n\neq 0} \frac{(2\pi i n)^k}{n^3} e_n(x)$ conv abs & unif by M-test

(b) We see

$n \neq 0$

$$\frac{d}{dx}\left(\overbrace{\frac{1}{n^3} e_n(x)}^{f_0}\right) = \frac{d}{dx}\left(\frac{1}{n^3} e^{2\pi i n x}\right)$$

$$= \overbrace{\frac{2\pi i n}{n^3} e^{2\pi i n x}}^{f_1}$$

so, b/c f_0, f_1 conv abs & unif, term-by-term works, and

$$\frac{d}{dx}\Big(\underbrace{\sum_{n\neq 0} \frac{1}{n^3} e_n(x)}_{f_0}\Big) = \underbrace{\sum_{n\neq 0} \frac{2\pi i n}{n^3} e_n(x)}_{f_1}.$$

$f \in C^1(S^1)$

12. (16 points) **PROOF QUESTION.** Prove that for $n \in \mathbf{Z}$, we have that

$$(e_n * f)(x) = \hat{f}(n)e_n(x).$$

$$(e_n * f)(x) = \int_0^1 e_n(x-t) f(t)\, dt$$

defn $e_n * f$

$$= \int_0^1 e^{2\pi i n(x-t)} f(t)\, dt$$

$$= \int_0^1 e^{2\pi i n x} e^{-2\pi i n t} f(t)\, dt$$

$$= e^{2\pi i n x} \int_0^1 f(t) \underbrace{e^{-2\pi i n t}}_{\overline{e_n(t)}}\, dt$$

$$= e^{2\pi i n x} \langle f, e_n \rangle = e^{2\pi i n x} \hat{f}(n)$$

$(t \in \mathbb{R})$

13. (16 points) **PROOF QUESTION.** For $f \in L^2(S^1)$, define $u : S^1 \times (0, +\infty) \to \mathbf{C}$ and $h : (0, +\infty) \to \mathbf{C}$ by

$$u(x,t) = \sum_{n \in \mathbf{Z}} \left(\frac{1}{t^2+1} \right) \hat{f}(n) e_n(x),$$
$$h(t) = \|u(x,t)\|^2 ,$$

Motivated by DEs

where the norm $\|u(x,t)\|$ is computed in $L^2_x(S^1)$, holding t constant.

M-test

(a) Use Parseval to prove that $h(t)$ is equal to a function series in t.

(b) Prove that $h(t)$ converges absolutely and uniformly to a continuous function.

(a) $$\underbrace{\|u(x,t)\|^2 = \sum_{n\in\mathbb{Z}} \left| \left(\frac{1}{t^2+1}\right) \hat{f}(n) \right|^2}_{\text{fn series in } t} \quad \text{(Parseval)}$$

(b) $\boxed{\begin{array}{l} \text{B/c } f \in L^2(S^1),\ \exists K \text{ s.t. } |\hat{f}(n)| \le K_1 \text{ for } n \in \mathbb{Z}. \\ \left(\text{Why: } |\hat{f}(n)| = \left|\int_0^1 f(x)\overline{e_n(x)}\,dx\right| \right. \\ \left. \le \int_0^1 |f(x)\underbrace{\overline{e_n(x)}}_{=1}|\,dx = \int_0^1 |f(x)|\,dx = K_1 \right) \end{array}}$

oops not needed!

So $$\left| \frac{1}{t^2+1} \hat{f}(n) \right|^2 \le |\hat{f}(n)|^2 = M_n$$

$$\sum_{n\in\mathbb{Z}} M_n = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \|f\|^2 \text{ convs by Parseval}$$

So $h(t) = \sum_{n\in\mathbb{Z}} \left|\frac{1}{t^2+1}\hat{f}(n)\right|^2$ convs abs & unif by M-test

Each $\left|\frac{1}{t^2+1}\hat{f}(n)\right|^2$ is cont.

so, since $\sum_{n\in\mathbb{Z}} \left|\frac{1}{t^2+1}\hat{f}(n)\right|^2$ convs $\boxed{\text{unif}}$

to $h(t)$, $h(t)$ is also cont.