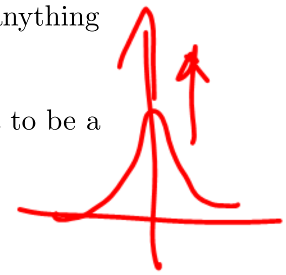
Math 131B, Fall 2017 **Name:** ________________________
Exam 3

This test consists of 9 questions on 7 pages, totalling $\boxed{114}$ points. You are not allowed to use books, notes, or calculators. Unless otherwise stated, you may take as given anything that has been proven in class, in the homework, or in the reading.



1. (12 points) Define what it means for a sequence of functions $K_n : [-\frac{1}{2}, \frac{1}{2}] \to \mathbf{R}$ to be a Dirac kernel.

1. (≥ 0) $K_n(t) \ge 0 \ \forall t \in [-\frac{1}{2}, \frac{1}{2}]$

2. (Area 1) $\int_{-1/2}^{1/2} K_n(t)\,dt = 1$

3. (Conc at 0) For $\delta > 0$

$$\lim_{n\to\infty} \int_{|t|\ge\delta} K_n(t)\,dt = 0.$$

Key: $f \in C^0$
$\lim_{n\to\infty} f * K_n = f$
unif.

In questions 2–4, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

2. (12 points) **TRUE/FALSE:** Let $f \in C^0(S^1)$ be a function such that

$$\hat{f}(n) = \begin{cases} \dfrac{1}{\sqrt{2^n}} & \text{for } n \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

$\int$ on S^1

Then it must be the case that $\int_0^1 |f(x)|^2\,dx = 1.$

$\int$ on $\mathbb{Z}$

By Isom Thm, we can compute ||f|| in terms of the Fourier coefficients of f.

$\sqrt{\quad}$ $\|f\|^2$

$$\|f\|^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2^n}}\right)^2 = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$\boxed{\text{TRUE}}$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

3. (12 points) **TRUE/FALSE:** Let V be an inner product space, and let $\mathcal{B} = \{u_1, \ldots, u_{13}\}$ be an orthogonal set of nonzero vectors in V. Then it is possible that there exist $c_1, \ldots, c_{13} \in \mathbf{C}$ such that

$$\left\| f - \overbrace{\sum_{n=1}^{13} \underbrace{\frac{\langle f, u_n \rangle}{\langle u_n, u_n \rangle}}_{\hat{f}(n)} u_n}^{\mathrm{proj}_{\mathcal{B}} f} \right\| = 5 \qquad \text{and} \qquad \left\| f - \underbrace{\sum_{n=1}^{13} c_n u_n}_{\text{also in span } \mathcal{B}} \right\| = 3.$$

FALSE: By the Best Approximation Theorem, the projection of f onto the span of B is the vector in span B that is closest to f. So this vector can't actually be closer to f than the projection of f onto the span of B.

4. (12 points) **TRUE/FALSE:** Let $f, g \in L^2(S^1)$ satisfy the property that $\hat{f}(n) = \hat{g}(n)$ for all $n \in \mathbf{Z}$. Then it is possible that $\|f - g\| = 2$.

FALSE. The Inversion Theorem (or Cor to it) says that if f and g have the same Fourier coefficients, then f = g almost everywhere (except on a set of measure zero; see Sec 7.4). But if f = g almost everywhere, then ||f - g|| = 0.

5. (12 points) **PROOF QUESTION.** It is a fact (i.e., you may take it as given) that if $F : [0,1] \times [0,1] \to \mathbf{C}$ is a continuous function such that $\dfrac{\partial F}{\partial x}$ is continuous on $[0,1] \times [0,1]$, then for all $x \in [a,b]$,

$$\frac{\partial}{\partial x} \int_0^1 F(x,y)\,dy = \int_0^1 \frac{\partial F}{\partial x}\,dy.$$

In other words, you may switch the order of the operations of differentiation in one variable and integration in the other.

Prove that for $f \in C^1(S^1)$ and $g \in C^0(S^1)$, we have that

$$\frac{d}{dx}((f*g)(x)) = \left(\frac{df}{dx} * g\right)(x).$$

(In other words, prove that the derivative of the convolution $f*g$ is equal to the convolution of $\dfrac{df}{dx}$ and g.)

$$(f*g)(x) = \int_0^1 f(x-t)\,g(t)\,dt$$

sum is x (outside var)

$$\frac{d}{dx}((f*g)(x)) = \frac{d}{dx}\int_0^1 f(x-t)\,g(t)\,dt$$

$$= \int_0^1 \frac{\partial}{\partial x}\big(f(x-t)\,g(t)\big)\,dt$$

const w.r.t. x

$$= \int_0^1 f'(x-t)\,g(t)\,dt$$

$$= \left(\frac{df}{dx} * g\right)(x)$$

6. (12 points) **PROOF QUESTION.** Let V be an inner product space, let $\{u_n \mid n \in \mathbf{N}\}$ be an orthogonal set, and suppose that for all $n \in \mathbf{N}$, u_n is orthogonal to g. Prove that for any coefficients $c_n \in \mathbf{C}$, if $\sum_{n=1}^{\infty} c_n u_n$ converges in the inner product metric in V, then

$$\left\langle \sum_{n=1}^{\infty} c_n u_n, g \right\rangle = 0.$$

$$\left\langle \sum_{n=1}^{\infty} c_n u_n, g \right\rangle \overset{\text{cont of IP}}{=} \sum_{n=1}^{\infty} \langle c_n u_n, g \rangle$$

$$\overset{\text{Defn of IP}}{=} \sum_{n=1}^{\infty} c_n \langle u_n, g \rangle$$

$$\overset{u_n \text{ orth to } g}{=} \sum_{n=1}^{\infty} c_n \cdot 0$$

$$= 0.$$

7. (14 points) **PROOF QUESTION.**

(a) Suppose f and g are continuous on S^1 and $f = g$ almost everywhere in S^1. What can we conclude about f and g?

(b) Suppose that $f \in \mathbf{C}^0(S^1)$ is such that for all $n \neq 0$, we have

M-test

$$\left|\hat{f}(n)\right| \leq \frac{1}{|n|^{3/2}}.$$

Prove that the Fourier series of f converges absolutely and uniformly to f.

(a) Then $f(x) = g(x)$ everywhere.

(b) Consider $\sum_{n \in \mathbb{Z}} \hat{f}(n) e_n(x)$. $e_n(x) = e^{2\pi i n x}$

$n \neq 0$ $\quad |\hat{f}(n) e_n(x)| = |\hat{f}(n)| \underbrace{|e_n(x)|}_{=1} = |\hat{f}(n)| \leq \frac{1}{|n|^{3/2}} = M_n$

Also $\sum_{n \neq 0} \frac{1}{|n|^{3/2}}$ convs by p-series ($p = 3/2 > 1$)

So by M-test, $\sum_{n \in \mathbb{Z}} \hat{f}(n) e_n(x)$ convs abs & unif to some g

g cont b/c unif lim of cont fns.

And $\hat{g}(n) = \left\langle \sum_{k \in \mathbb{Z}} \hat{f}(k) e_k(x), e_n(x) \right\rangle$ (cont IP)

$= \sum_{k \in \mathbb{Z}} \hat{f}(k) \underbrace{\langle e_k(x), e_n(x) \rangle}_{=0 \text{ unless } k=n} = \hat{f}(n) \langle e_n(x), e_n(x) \rangle = \hat{f}(n)$ $\forall n \in \mathbb{Z}$

So $g = f$ a.e. $\Rightarrow \boxed{f = g}$

8. (14 points) **PROOF QUESTION.** Let $\mathcal{H}$ be a Hilbert space, $\{e_n \mid n \in \mathbf{N}\}$ an ortho**normal** set in $\mathcal{H}$, and $c_n \in \mathbf{C}$.

(a) State the Hilbert Space Absolute Convergence Theorem for $\displaystyle\sum_{n=1}^{\infty} c_n e_n$.

(b) Now suppose that for all $n \in \mathbf{N}$,

$$|c_n| \le \frac{1}{n^{2/3}}.$$

Prove that $\displaystyle\sum_{n=1}^{\infty} c_n e_n$ converges (in the inner product metric) to some $f \in \mathcal{H}$.

$\|e_n\|^2 = 1$

(a) HSACT $\displaystyle\sum_{n=1}^{\infty} c_n e_n$ convs in $\mathcal{H}$ $\iff \displaystyle\sum_{n=1}^{\infty} |c_n|^2$ convs in $\mathbb{R}$.

(b) Note

$$|c_n|^2 \le \left|\frac{1}{n^{2/3}}\right|^2 = \frac{1}{|n|^{4/3}}$$

Also $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ convs by p-series ($p = 4/3 > 1$)

So by comparison test, $\displaystyle\sum_{n=1}^{\infty} |c_n|^2$ convs.

HSACT $\Rightarrow \displaystyle\sum_{n=1}^{\infty} c_n e_n$ convs to $\mathcal{H}$.

9. (14 points) **PROOF QUESTION.** Let V and W be normed spaces, and let $T : V \to W$ be a function such that $T(0) = 0$ and

$$\|T(f)\| \leq 13\,\|f\|$$

for all $f \in V$. Prove that T is continuous at 0.

$\boxed{7.2}$

Defn T cont at 0: $\forall \epsilon > 0$ $\exists \delta(\epsilon)$ s.t. if $\|f - 0\| < \delta(\epsilon)$ then $\|T(f) - T(0)\| < \epsilon$.

(A) $\epsilon > 0$

Pick $\delta(\epsilon) = \frac{\epsilon}{13}$

(A) $\|f - 0\| < \delta(\epsilon) = \frac{\epsilon}{13}$

$$\|T(f) - T(0)\| = \|T(f) - 0\| = \|T(f)\| \leq 13\,\|f\| < 13\left(\frac{\epsilon}{13}\right) = \epsilon$$

(C) $\|T(f) - T(0)\| < \epsilon$