Math 131B, Fall 2017 **Name:** ________________________

Exam 2

This test consists of 8 questions on 7 pages, totalling 100 points. You are not allowed to use books, notes, or calculators. Unless otherwise stated, you may take as given anything that has been proven in class, in the homework, or in the reading. You may also freely use the following integrals and other formulas arising from $e_n(x) = e^{2\pi i n x}$:

$$\int \overline{e_n(x)}\,dx = -\frac{e_{-n}(x)}{2\pi i n} + C$$

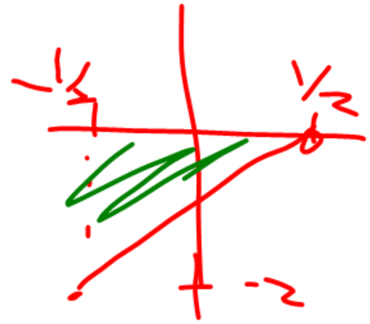
$$\int x\,\overline{e_n(x)}\,dx = -\frac{xe_{-n}(x)}{2\pi i n} - \frac{e_{-n}(x)}{(2\pi i n)^2} + C$$

$$\int x^2\,\overline{e_n(x)}\,dx = -\frac{x^2e_{-n}(x)}{2\pi i n} - \frac{2xe_{-n}(x)}{(2\pi i n)^2} - \frac{2e_{-n}(x)}{(2\pi i n)^3} + C$$

$$\int_0^1 e_n(x)\,\overline{e_k(x)}\,dx = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

$$e_n(k) = e_{-n}(k) = 1 \qquad e_n\left(\frac{1}{2}\right) = e_{-n}\left(\frac{1}{2}\right) = (-1)^n$$

$$e_n\left(\frac{1}{4}\right) = e_{-n}\left(-\frac{1}{4}\right) = i^n \qquad e_n\left(-\frac{1}{4}\right) = e_{-n}\left(\frac{1}{4}\right) = (-i)^n$$



1. (14 points) Let $f : S^1 \to \mathbf{C}$ be given by

$$f(x) = 2x - 1 \qquad \text{for } -\frac{1}{2} \le x < \frac{1}{2}.$$

Calculate the Fourier coefficients $\hat{f}(n)$ ($n \in \mathbf{Z}$). Show all your work, and do not simplify your final answers.

$$\hat{f}(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\,\overline{e_n(x)}\,dx$$

$\boxed{n=0}$

$$\hat{f}(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\,dx = -1 \quad \text{b/c}$$

$\boxed{n \neq 0}$

$$\hat{f}(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} (2x-1)\,\overline{e_n(x)}\,dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} 2x\,\overline{e_n(x)}\,dx - \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{e_n(x)}\,dx$$

$$= 2\left(-\frac{x\,e_{-n}(x)}{2\pi i n} - \frac{e_{-n}(x)}{(2\pi i n)^2}\right)\Bigg]_{-\frac{1}{2}}^{\frac{1}{2}} - \frac{e_{-n}(x)}{(-2\pi i n)}\Bigg]_{-\frac{1}{2}}^{\frac{1}{2}}$$

period 1, so cancel on eval

$$= 2\left(-\frac{\frac{1}{2}\,e_{-n}(\frac{1}{2})}{2\pi i n} + \frac{-\frac{1}{2}\,e_{-n}(-\frac{1}{2})}{2\pi i n}\right)$$

$$= 2\left(-\frac{(-1)^n}{4\pi i n} - \frac{(-1)^n}{4\pi i n}\right) = -\frac{(-1)^n}{\pi i n}$$

2. (14 points) Let V be an inner product space.

(a) State the Cauchy-Schwarz inequality for $f, g \in V$.

(b) State the Triangle inequality for $f, g \in V$.

(a) $|\langle f, g\rangle| \leq \|f\| \, \|g\|$

(b) $\|f+g\| \leq \|f\| + \|g\|$

In questions 3–5, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

3. (12 points) **TRUE/FALSE:** It is possible that $f \in C^2(S^1)$ and that $\hat{f}(n) = \dfrac{7}{n}$ for $n \neq 0$.

False: For $n \neq 0$: $|\hat{f}(n)| \leq \dfrac{K_2}{n^2}$

But $\dfrac{7}{n}$ is not $\leq \dfrac{K_2}{n^2}$ for any K_2.

$\dfrac{100000}{n^2} \ll \dfrac{7}{n}$

$\to 0$ faster.

$\dfrac{7}{n}$ vs $\dfrac{7}{n^2}$ $\quad$ $\dfrac{7}{10}$ vs $\dfrac{7}{100}$

I.e., as n -> infinity, 100000/n^2 will be much less than 7/n, since lim of (100000/n^2)/(7/n) = 0.

4. (12 points) **TRUE/FALSE:** Let X be a nonempty open subset of $\mathbf{C}$. If $f_n : X \to \mathbf{C}$ is a sequence of differentiable functions that converges pointwise to some $f : X \to \mathbf{C}$, and $f_n' : X \to \mathbf{C}$ is a sequence of continuous functions that converges uniformly to some $g : X \to \mathbf{C}$, then it must be the case that f is differentiable and $f' = g$.

True (Just need $f_n' \to g$ unif)

Review 6 NOs

Unif conv $\Rightarrow$ YES

5. (12 points) **TRUE/FALSE:** If $f_n : [0,1] \to \mathbf{C}$ is a sequence of continuous functions that converges pointwise to some $f : [0,1] \to \mathbf{C}$, then it must be the case that f is continuous.

False

6 NOs

$f_n(x) = x^n$

$$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

Then $\lim_{n \to \infty} x^n = f(x)$, f_n cont, f not cont.

6. (10 points) **PROOF QUESTION.** Let

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n.$$

(a) Find the radius of convergence R of $f(x)$, with justification. (If you don't remember how to do this, continue to the rest of the problem and just use R as an unknown constant in your answer.)

(b) For which $x \in \mathbf{R}$ is term-by-term differentiation valid?

(c) Use term-by-term differentiation to prove that

$$f'(x) - xf'(x) = 1$$

for all values of x listed in part (b). (Suggestion: You may find the substitution $k = n - 1$ to be useful.)

(a) $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n\to\infty} \dfrac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)} = \lim\limits_{n\to\infty} \dfrac{n}{n+1}$

$= \lim\limits_{n\to\infty} \dfrac{1}{1+\frac{1}{n}}$ $\left(\frac{1}{n} \to 0\right)$ $= 1$. So ROC $= \frac{1}{1} = 1$.

$\boxed{R=1}$

(b) $|x| < R = 1$

$k = n-1$

(c) $f'(x) = \dfrac{d}{dx} \sum\limits_{n=1}^{\infty} \dfrac{1}{n} x^n = \sum\limits_{n=1}^{\infty} \dfrac{1}{n}(nx^{n-1}) = \sum\limits_{n=1}^{\infty} x^{n-1}$

$xf'(x) = \sum\limits_{n=1}^{\infty} x^n$ same except x^0 term $= \sum\limits_{k=0}^{\infty} x^k$

$f'(x) - xf'(x) = \sum\limits_{k=0}^{\infty} x^k - \sum\limits_{k=1}^{\infty} x^k =$ "x^0" $= 1$.

7. (12 points) **PROOF QUESTION.** Consider the function space $V = C^0([a,b])$ ($a < b$ in $\mathbf{R}$), and define the inner product

$$\langle f(x), g(x)\rangle = \int_a^b f(x)\overline{g(x)}\,dx$$

on V. Suppose $\{p_n(x) \mid n \geq 0\}$ is a set of polynomial functions such that

$$\int_a^b p_i(x)\overline{p_j(x)}\,dx = \quad \langle p_i(x), p_j(x)\rangle = \begin{cases} 0 & \text{if } i \neq j, \\ 7 & \text{if } i = j. \end{cases}$$

orthog

Now suppose $c_n \in \mathbf{C}$ is a choice of coefficients such that $\sum_{n=0}^{\infty} c_n p_n(x)$ converges absolutely and uniformly to some function $f(x)$. Prove that for $k \geq 0$, we have

$$\int_a^b f(x)\overline{p_k(x)}\,dx = 7c_k.$$

Make sure to justify all steps carefully.

$$\int_a^b f(x)\overline{p_k(x)}\,dx = \int_a^b \sum_{n=0}^{\infty} c_n p_n(x)\overline{p_k(x)}\,dx$$

$$= \sum_{n=0}^{\infty} \underbrace{\int_a^b c_n p_n(x)\overline{p_k(x)}\,dx}$$

Unif conv of series means we can swap sum and integral.

Given fact about <p_i,p_ j> means that this integral is =0 unless n=k. So all terms drop out of the sum except the n=k term.

$$= c_k \underbrace{\int_a^b p_k(x)\overline{p_k(x)}\,dx}_{=7} = 7c_k.$$