Sample 2020

Math 131B, ~~Fall 2019~~ **Name:** ______________________
Exam 1

This test consists of 8 questions on 6 pages, totalling 100 points. You are not allowed to use books, notes, or calculators. Unless otherwise stated, you may take as given anything that has been proven in class, in the homework, or in the reading.

1. (12 points) The goal of this problem is to define the Riemann integral.

Let $v : [a, b] \to \mathbf{R}$ be a bounded real-valued function. Recall that a *partition* P of $[a, b]$ is a finite subset $\{x_0, \ldots, x_n\} \subset [a, b]$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, let $(\Delta x)_i = x_i - x_{i-1}$, and define

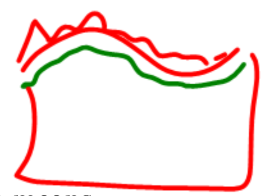
$$M(v; P, i) = \sup\{v(x) \mid x \in [x_{i-1}, x_i]\} \qquad m(v; P, i) = \inf\{v(x) \mid x \in [x_{i-1}, x_i]\}$$

$$U(v; P) = \sum_{i=1}^{n} M(v; P, i)(\Delta x)_i \qquad L(v; P) = \sum_{i=1}^{n} m(v; P, i)(\Delta x)_i$$

$\mathcal{P}$

Let $\mathcal{P}$ be the set of all partitions of $[a, b]$. (I.e., the beginning of the definition of the Riemann integral has been given to you.)

(a) Define the upper and lower Riemann integrals $\overline{\int_a^b} v(x)\,dx$ and $\underline{\int_a^b} v(x)\,dx$.



(b) Define what it means for v to be integrable on $[a, b]$, and define $\int_a^b v(x)\,dx$.

(c) Now let $f : [a, b] \to \mathbf{C}$ be a bounded complex-valued function. Define what it means for f to be integrable on $[a, b]$, and define $\int_a^b f(x)\,dx$.

(a) $\overline{\int_a^b} v(x)\,dx = \inf\{U(v;P) \mid P \in \mathcal{P}\}$

$\underline{\int_a^b} v(x)\,dx = \sup\{L(v;P) \mid P \in \mathcal{P}\}$

(b) Means $\overline{\int_a} v\,dx = \underline{\int_a} v\,dx$; then both $\int_a^b v\,dx$.

(c) If u, v are Re, Im part of $f = u + iv$, then $\int_a^b f(x)\,dx = \int_a^b u(x)\,dx + i\int_a^b v(x)\,dx$

f $\int$-ble when u, v $\int$-ble, & then

both

2. (12 points) Let X be a nonempty subset of $\mathbf{C}$, let $a \in X$, and let $f : X \to \mathbf{C}$ be a function. Give the ϵ-δ definition of what it means for f to be continuous at a.

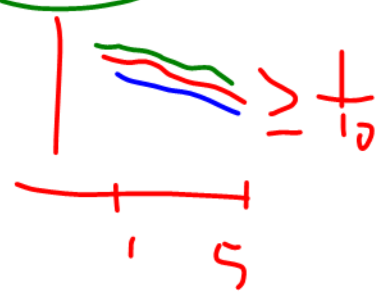
$\forall \epsilon > 0$
$\exists \delta(\epsilon) > 0$ s.t.
If $|x - a| < \delta(\epsilon)$
then $|f(x) - f(a)| < \epsilon$

In questions 3–5, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

False = need some kind of counterproof

3. (12 points) **TRUE/FALSE:** Let $v : [1, 5] \to \mathbf{R}$ be a function such that for every partition P of $[1, 5]$, we have that $U(v; P) - L(v; P) \geq \frac{1}{10}$. Then it is possible that v is integrable on $[1, 5]$.

$\boxed{\text{False}}$ Seq crit for $\int$:



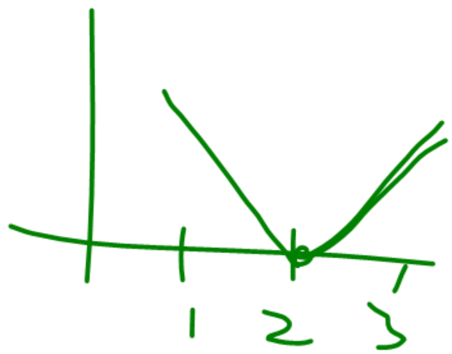
v $\int$ble on $[1,5]$
$\Updownarrow$
$\forall \epsilon > 0, \exists P$ s.t. $U(v;P) - L(v;P) < \epsilon$

This fails for $\epsilon = \frac{1}{100}$.

fnct diff at 2

4. (12 points) **TRUE/FALSE:** There exists a continuous function $f : [1, 3] \to \mathbf{C}$ such that the limit $\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$ does not exist.

$\boxed{\text{TRUE}}$ Ex:



optional

$f(y) = |x - 2|$

5. (12 points) **TRUE/FALSE:** Let S be a nonempty subset of $\mathbf{R}$ such that for all $x \in S$, $x \leq \frac{31}{8}$. Then it is possible that $\sup S = 4$.

FALSE sup S is least u.b. of S so 4 can't be sup b/c $\frac{31}{8}$ is an u.b. for S and $\frac{31}{8} < 4$.

6. (12 points) **PROOF QUESTION.** Let $v : [-2, 5] \to \mathbf{R}$ be a bounded function, and suppose that:

A_1 • For every $\epsilon > 0$, there exists a partition P_1 of $[-2, 3]$ such that $|7 - L(v; P_1)| < \epsilon$; and

A_2 • For every $\epsilon > 0$, there exists a partition P_2 of $[3, 5]$ such that $|6 - L(v; P_2)| < \epsilon$.

Prove that for any $\epsilon > 0$, there exists a partition Q of $[-2, 5]$ such that $|13 - L(v; Q)| < \epsilon$. (Suggestion: You may want to draw a picture of the situation.)

$\forall \epsilon > 0$

By A_1, $\exists P_1$ of $[-2,3]$ s.t. $|7 - L(v;P_1)| < \frac{\epsilon}{2}$.

By A_2, $\exists P_2$ of $[3,5]$ st. $|6 - L(v;P_2)| < \frac{\epsilon}{2}$

Let $Q = P_1 \cup P_2$

So $L(v;Q) = L(v;P_1) + L(v;P_2)$ (by picture)

So $|13 - L(v;Q)| = |13 - (L(v;P_1) + L(v;P_2))|$

$= |(7 - L(v;P_1)) + (6 - L(v;P_2))|$

$\triangle$ $\leq |7 - L(v;P_1)| + |6 - L(v;P_2)| = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

C $|13 - L(v;Q)| < \epsilon$

Outline

7. (14 points) **PROOF QUESTION.** Let X be a nonempty subset of $\mathbf{C}$, let a be a point of X, and suppose $f : X \to \mathbf{C}$ is a function with the following property:

- For any sequence x_n in X such that $\lim_{n\to\infty} x_n = a$, we have that $\lim_{n\to\infty} f(x_n) = f(a)$. (*)

Let $g : X \to \mathbf{C}$ be defined by $g(x) = (3 + 4i)f(x)$. Use the definition of continuity of g (in any version) to prove that g is continuous at a. You may use the limit laws for sequences, but you may not use the laws of continuity, since the goal of this problem is to re-prove a special case of one of the laws of continuity.

f cont at a

CM of f

Seq cont of g

(A) x_n in X, $\lim_{n\to\infty} x_n = a$

Then $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} (3+4i) f(x_n)$

$= (3+4i) \lim_{n\to\infty} f(x_n)$ CM law for seqs.

$= (3+4i) f(a)$ by (*)

$= g(a)$.

(C) $\lim_{n\to\infty} g(x_n) = g(a)$

ALT

B/c f cont at a,

$\forall \epsilon_0 > 0, \exists \delta_0(\epsilon_0) > 0$ s.t. if $|x-a| < \delta_0(\epsilon_0)$ then $|f(x)-f(a)| < \epsilon_0$; (**)

(A) $\epsilon > 0$

Let $\delta(\epsilon) = \delta_0(\frac{\epsilon}{5})$ $\left(\epsilon_0 = \frac{\epsilon}{5}\right)$

(A) $x \in X$, $|x-a| < \delta(\epsilon) = \delta_0(\frac{\epsilon}{5})$

Want

$|g(x)-g(a)| < \epsilon$

$= |(3+4i)f(x) - (3+4i)f(a)|$

$= |3+4i|\,|f(x)-f(a)|$

$= 5|f(x)-f(a)|$

$|f(x)-f(a)| < \frac{\epsilon}{5}$

$5 \big)\, 4$
3

So by (**),

$|f(x)-f(a)| < \epsilon_0 = \frac{\epsilon}{5}$

$5|f(x)-f(a)| < \epsilon$

$\|$

$|3+4i|\,|f(x)-f(a)|$

$\|$

by $\vdots$

$\|$

$|g(x)-g(a)|$

(C) $|g(x)-g(a)| < \epsilon$

(C) g cont at a

8. (14 points) **PROOF QUESTION.** Let X be a metric space and let L be a point in X.

(a) Let x_n be a sequence in X. Define what it means to say that $\lim_{n\to\infty} x_n = L$.

(b) Now let x_n and y_n be sequences in X such that $\lim_{n\to\infty} y_n = L$ and for all n,

$$d(L, x_n) \leq 17 d(L, y_n).$$

Use the definition from part (a) to prove that $\lim_{n\to\infty} x_n = L$. (In particular, do not just quote the Squeeze Lemma, because you are re-proving a special case of the Squeeze Lemma.)

(a) $\forall \epsilon > 0, \exists N(\epsilon)$ s.t. if $n \in \mathbb{Z}, n > N(\epsilon)$, then $d(x_n, L) < \epsilon$.

(b) A $\forall \epsilon_0 > 0, \exists N_0(\epsilon_0)$ s.t. if $n \in \mathbb{Z}, n > N_0(\epsilon_0)$, then $d(y_n, L) < \epsilon_0$. $(*)$

A $d(x_n, L) \leq 17 d(y_n, L)$ $(**)$

A $\epsilon > 0$ $\quad \epsilon_0 = \frac{\epsilon}{17}$

Pick $N(\epsilon) = N_0\left(\frac{\epsilon}{17}\right)$

A $n > N(\epsilon) = N_0\left(\frac{\epsilon}{17}\right)$

By $(*)$, $d(y_n, L) < \epsilon/17$

So $d(x_n, L) \leq 17 d(y_n, L) < 17 \frac{\epsilon}{17} = \epsilon$

C $d(x_n, L) < \epsilon$

C $\lim_{n\to\infty} x_n = L$

Want $d(x_n, L) \leq 17 d(y_n, L) < \epsilon$

$d(y_n, L) < \frac{\epsilon}{17}$

$\epsilon_0 = \epsilon/17$