Math 131B, Fall 2023 **Name:** ________________________

Exam 3

This test consists of 8 questions on 8 pages, totalling 100 points. You are not allowed to use books, notes, or calculators. Unless otherwise stated, you may take as given anything that has been proven in class, in the homework, or in the reading.

1. (10 points) For each of the following classes of functions f, state the best results that we have seen about the convergence of the Fourier series of f. In particular, when you state a convergence result, be careful to specify the **type** of convergence in question.

(a) $f \in L^2(S^1)$.

(b) $f \in C^0(S^1)$. (Note that the best result that applies to *all* continuous f is not actually about the Fourier series of f, per se.)

(c) $f \in C^1(S^1)$.

In questions 2–4, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

2. (12 points) **TRUE/FALSE:** Let $\{u_n \mid n \in \mathbf{N}\}$ be a set of nonzero orthogonal vectors in a Hilbert space $\mathcal{H}$ with the property that:

For any $f \in \mathcal{H}$, if $\langle f, u_n \rangle = 0$ for all $n \in \mathbf{N}$, then $f = 0$.

Then it is possible that there exists some $f \in \mathcal{H}$ such that the generalized Fourier series $\displaystyle\sum_{n=1}^{\infty} \hat{f}(n) u_n$ does *not* converge to f in the inner product metric.

3. (12 points) **TRUE/FALSE:** Let $f, g \in C^0(S^1)$ be functions such that $\hat{g}(2k) = 0$ for all $k \in \mathbf{Z}$. Then it is possible that $\widehat{(f * g)}(6) = 3 + 2i$.

4. (12 points) **TRUE/FALSE:** For any $f \in L^2(S^1)$ and $c_{-8}, \ldots, c_0, \ldots, c_8 \in \mathbf{C}$, it must be the case that

$$\|f - f_8\| \leq \left\| f - \sum_{n=-8}^{8} c_n e_n \right\|,$$

where f_8 is the Fourier polynomial of f of degree 8.

$e_n(x+y) = e_n(x)e_n(y)$

5. (13 points) **PROOF QUESTION.** Let $f : S^1 \to \mathbf{C}$ be continuous, and recall that $e_n(x) = e^{2\pi inx}$. Let

$$G_N(x) = \sum_{k=0}^{N} e_{2k}(x).$$

Use the **definition** of convolution to prove that

sum = x

$$(G_N * f)(x) = \sum_{k=0}^{N} \hat{f}(2k)e_{2k}(x).$$

$$(G_N * f)(x) = \int_0^1 G_N(x-t)f(t)\,dt$$

$$= \int_0^1 \sum_{k=0}^{N} e_{2k}(x-t)f(t)\,dt$$

prop of $\int$

$$= \sum_{k=0}^{N} \int_0^1 e_{2k}(x-t)f(t)\,dt$$

$$= \sum_{k=0}^{N} \int_0^1 e_{2k}(x)e_{2k}(-t)f(t)\,dt$$

$$= \sum_{k=0}^{N} e_{2k}(x) \int_0^1 f(t)\overline{e_{2k}(t)}\,dt$$

$$= \sum_{k=0}^{N} e_{2k}(x)\hat{f}(2k)$$

6. (13 points) **PROOF QUESTION.** Let $\mathcal{H}$ be a Hilbert space, let $\{u_n \mid n \in \mathbf{N}\}$ be an orthogonal set, and suppose that $c_n \in \mathbf{C}$ are such that $f = \sum_{n=1}^{\infty} c_n u_n$ converges in the IP metric in $\mathcal{H}$.

(a) Define $f = \sum_{n=1}^{\infty} c_n u_n$ as a limit of finite sums.

$$f = \lim_{N\to\infty} \sum_{n=1}^{N} c_n u_n$$

(b) Prove that for fixed $k \geq 1$, we have that

$$\langle f, u_k \rangle = c_k \|u_k\|^2 .$$

Briefly **JUSTIFY** each step, and in particular, each time you pull out a limit or an infinite sum, make sure to **JUSTIFY** that operation carefully.

$$\langle f, u_k \rangle = \left\langle \lim_{N\to\infty} \sum_{n=1}^{N} c_n u_n, u_k \right\rangle$$

$$= \lim_{N\to\infty} \left\langle \sum_{n=1}^{N} c_n u_n, u_k \right\rangle$$

IP cont in each var

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \langle c_n u_n, u_k \rangle$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} c_n \langle u_n, u_k \rangle$$

Defn IP

$$= \sum_{n=1}^{\infty} c_n \langle u_n, u_k \rangle$$

Orth: If $n \neq k$, $\langle u_n, u_k \rangle = 0$

$$= c_k \langle u_k, u_k \rangle$$

$$= c_k \|u_k\|^2$$

7. (14 points) **PROOF QUESTION.** Let $f : S^1 \to \mathbf{R}$ be integrable, with

$$\int_{-1/2}^{1/2} |f(t)|\, dt = 3,$$

and let $h(x,t)$ be a real-valued function in two variables $x, t \in S^1$ such that $h(x,t)$ is continuous in each variable. Suppose h also satisfies the following condition:

> *Condition on* $h(x,t)$. For any $\epsilon_1 > 0$, there exists some $\delta_1(\epsilon_1) > 0$ such that if $|x| < \delta_1(\epsilon_1)$, then $|h(x,t) - h(0,t)| < \epsilon_1$.

h cont at 0, ind of t

Let

$$g(x) = \int_{-1/2}^{1/2} h(x,t) f(t)\, dt.$$

Prove that for every $\epsilon > 0$, there exists some $\delta > 0$ such that if $|x| < \delta$, then $|g(x) - g(0)| < \epsilon$.

$\boxed{A}$ $\epsilon > 0$. Let $\epsilon_1 = \frac{\epsilon}{4}$.

Let $\delta = \delta_1(\epsilon_1) = \delta_1\left(\frac{\epsilon}{4}\right)$

A $|x| < \delta$. Then

$$|g(x) - g(0)| = \left| \int_{-1/2}^{1/2} h(x,t) f(t)\, dt - \int_{-1/2}^{1/2} h(0,t) f(t)\, dt \right|$$

$$= \left| \int_{-1/2}^{1/2} \big(h(x,t) - h(0,t)\big) f(t)\, dt \right|$$

$\downarrow$ $\triangle$ for $\int$

$$\leq \int_{-1/2}^{1/2} \underbrace{|h(x,t) - h(0,t)|}_{\text{control}}\, |f(t)|\, dt$$

$$\leq \int_{-1/2}^{1/2} \epsilon_1 |f(t)|\, dt$$

$$= \epsilon_1 \int_{-1/2}^{1/2} |f(t)|\, dt = 3\epsilon_1 = \frac{3\epsilon}{4} < \epsilon.$$

C $|g(x) - g(0)| < \epsilon$

8. (14 points) **PROOF QUESTION.** Let $\{e_n \mid n \in \mathbf{Z}\}$ be the usual basis for $L^2(S^1)$ and suppose $a_n \in \mathbf{C}$.

(a) State the Hilbert Space Absolute Convergence Theorem for $\sum_{n\in\mathbf{Z}} a_n e_n$.

(b) Now suppose that for all $n \in \mathbf{Z}$,

$$|a_n| \leq \frac{1}{|n|^{3/4}+5}.$$

Prove that $\sum_{n\in\mathbf{Z}} a_n e_n$ converges (in the inner product metric) to some $f \in L^2(S^1)$.

(a) $\sum_{n\in\mathbb{Z}} a_n e_n$ convs in IP $\iff$ $\sum_{n\in\mathbb{Z}} |a_n|^2 \|e_n\|^2$ convs

(b) $|a_n|^2 \leq \left(\frac{1}{|n|^{3/4}+5}\right)^2 \leq \frac{1}{|n|^{6/4}}$ $(n \neq 0)$

$\sum_{n\neq 0} \frac{1}{|n|^{6/4}}$ convs by p-series $(p = \frac{6}{4} > 1)$

So, b/c one term $(n=0)$ doesn't affect conv,

$\sum_{n\in\mathbb{Z}} |a_n|^2$ convs by comparison.

By HSACT, $\sum_{n\in\mathbb{Z}} a_n e_n$ convs in IP.

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