Math 131A, Spring 2024 **Name:** ________________________

Final exam

This test consists of 14 questions on 12 pages, totalling 200 points. You are not allowed to use books, notes, or calculators. Unless otherwise stated, you may take as given anything that has been proven in class, in the homework, or in the reading.

1. (16 points) Let $f : [a, b] \to \mathbf{R}$ be differentiable on (a, b) and continuous on $[a, b]$. State the Mean Value Theorem, as applied to f.

2. (15 points) Let S be a nonempty subset of $\mathbf{R}$.

(a) Define what it means for a real number L to be a lower bound for S.

(b) Define $\inf S$.

3. (15 points) Let $a_n = \dfrac{17n^2}{2^n}(3+5^{-n})$. Determine if the series $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{17n^2}{2^n}(3+5^{-n})$ converges or diverges, and prove your answer.

For questions 4–9, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

4. (13 points) **TRUE/FALSE:** Let a_n be a sequence such that $3 \leq a_n \leq 13$ for all $n \in \mathbf{N}$. Then it is possible that $\lim_{k\to\infty} a_{n_k}$ does not exist for any subsequence a_{n_k} of a_n.

5. (13 points) **TRUE/FALSE:** Let f be a continuous function on $\mathbf{R}$. Then it is possible that there exists a sequence x_n such that $\lim_{n\to\infty} x_n = 11$, $\lim_{n\to\infty} f(x_n) = 2$, and $f(11) = -5$.

6. (13 points) **TRUE/FALSE:** Let $f_n(x)$ be a sequence of differentiable functions that converges pointwise to $f(x)$ on $[0, 1]$. Then it must be the case that $f(x)$ is differentiable on $[0, 1]$.

7. (13 points) **TRUE/FALSE:** Let $f : [2, 9] \to \mathbf{R}$ be a differentiable function. Then it must be the case that

$$\underline{\int_2^9} f(x)\, dx = \overline{\int_2^9} f(x)\, dx.$$

8. (13 points) **TRUE/FALSE:** Let $f : [-5, 8] \to \mathbf{R}$ be a continuous function. Then it is possible that

$$f([-5, 8]) = \{f(x) \mid x \in [-5, 8]\} = [3, 4) = \{y \in \mathbf{R} \mid 3 \leq y < 4\}.$$

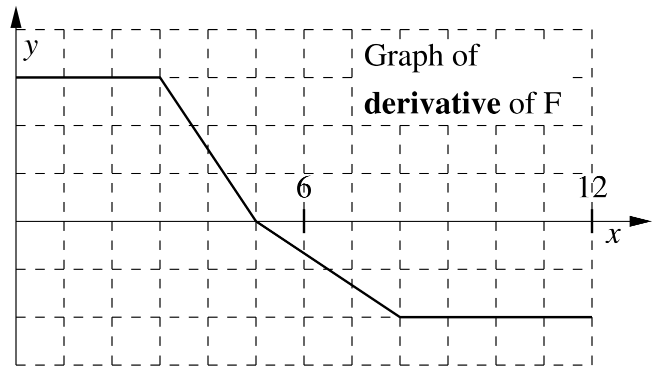
9. (13 points) **TRUE/FALSE:** Let a_n be a sequence such that $3 \leq a_n \leq 13$ for all $n \in \mathbf{N}$. Then it is possible that $\lim_{n\to\infty} a_n$ does not exist.

10. (15 points) **PROOF QUESTION.** Define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} x^2 + 2x & \text{if } x \in \mathbf{Q}, \\ x^{6/5} + 2x & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Prove that $f'(0) = 2$.

11. (15 points) **PROOF QUESTION.** Let $F : [0, 12] \to \mathbf{R}$ be a differentiable function such that $F(0) = -4$ and the **DERIVATIVE** F' of F has the following graph:



Assume that each box is 1×1 and anything that appears to be a straight line really is a straight line.

(a) Briefly **EXPLAIN** why (prove that) you can be sure that F attains an absolute (global) minimum at some $x \in [0, 12]$.

(b) At which value(s) of x does F attain an absolute (global) minimum on $[0, 12]$, and what is the actual absolute (global) minimum value of $y = F(x)$ on $[0, 12]$? **JUSTIFY** (prove) your answer.

12. (15 points) **PROOF QUESTION.** Suppose $\lim_{n\to\infty} a_n = 0$.

(a) Write out the definition (using epsilon) of $\lim_{n\to\infty} a_n = 0$.

(b) Let $b_n = 13a_n \sin(n^2+n)$. Use the definition of the limit of a sequence (not the Squeeze Theorem, etc.) to prove that $\lim_{n\to\infty} b_n = 0$.

13. (15 points) **PROOF QUESTION.** Define $g : \mathbf{R} \to \mathbf{R}$ by

$$g(x) = \begin{cases} x^{1/3} \sin\left(\dfrac{1}{x^2}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Use the ϵ-δ definition of continuity to prove that g is continuous at 0.

14. (16 points) **PROOF QUESTION.** Define $h : I \to \mathbf{R}$ by $h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2 5^n}$, where I is the interval of convergence of the series.

In the following, if you can't complete one subquestion, you can still use its result in the rest of the problem (i.e., you can use part (a) in parts (b) and (c), and part (b) in part (c)).

(a) Compute the radius of convergence R of h.

(b) Prove that h converges uniformly on $[-R, R]$. (Suggestion: Weierstrass M-test. For partial credit, prove that $h(x)$ converges for $x = \pm R$ without worrying about uniform convergence.)

(c) Must h be continuous on $[-R, R]$? Briefly explain why or why not.

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