In-class groupwork activities

Fourier series, Fourier transforms, and function spaces:
A second course in analysis

Teamwork roles:

- The **facilitator** organizes the team to make sure the task is complete and makes sure all team members stay on task.
- The **documenter** writes the team's work on the board.
- The **presenter** talks through the team's work to the entire class at the end.
- (if a team of four) The **verifier** makes sure that everyone on the team understands the team's final answer.

Ch. 1: Define $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Carefully "prove" that $\frac{d}{dx} e^x = e^x$. What theorem(s)/facts do you use in each step?

2.1: Write down the definition of $\sup S$.
If possible, find sets $S \subseteq \mathbf{R}$ such that:

- $\sup S = \max S$.
- $\sup S < \max S$.
- $\sup S > \max S$.
- $\sup S$ exists but $\max S$ doesn't.
- $u = \sup S$ is in S but *isolated*, i.e., there exists some $\epsilon > 0$ such that no point of S other than u is within ϵ of u.
- There exists a sequence of values in S whose limit is $\sup S$. (Informally; we haven't gotten to the definition of limit yet.)
- There does not exist a sequence of values in S whose limit is $\sup S$.

2.1: Let S be a nonempty bounded subset of $\mathbf{R}$ and $u, \ell \in \mathbf{R}$.

1. (Sup lemma) Prove that the following are equivalent:
   - $u \geq \sup S$.
   - u is an upper bound for S.
2. (Inf lemma) Prove that the following are equivalent:
   - $\ell \leq \inf S$.
   - ℓ is a lower bound for S.

2.3: Consider the following "distance functions" on $\mathbf{R}^2$.

- $d_1((x_1, y_1), (x_2, y_2)) = (x_1 - x_2)^2 + (y_1 - y_2)^2$
- $d_2((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
- $d_3((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$
- $d_4((x_1, y_1), (x_2, y_2)) = |x_1 - x_2|^{1/2} + |y_1 - y_2|^{1/2}$

For each d_i that is **not** a metric, find a counterexample to the triangle inequality.

2.4: Write down the definition of $\lim_{n\to\infty} a_n = L$.
Consider one of the following limit laws:

1. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then $\lim_{n\to\infty} a_n + b_n = L + M$.
2. If $\lim_{n\to\infty} a_n = L$ and $c \in \mathbf{C}$, then $\lim_{n\to\infty} ca_n = cL$.

Steps:

1. Write out the assumption and conclusion of the proof, at the highest level.
2. The conclusion of the proof is now a statement about a limit, which is in turn equivalent to an if-then statement. Write out the assumption-conclusion pair for that if-then statement, nested inside the assumption-conclusion pair from part **??**.
3. The inner conclusion from part **??** is a "there exists" statement. Expand that statement in a third level of nesting, putting a guess of the value of something at the beginning and a matching conclusion at the end.
4. Your conclusion from part **??** is again an if-then statement; create an inner assumption-conclusion pair to match that.
5. Now return to the original assumption from **??**, which is itself a statement or statements about limits. Expand that statement using the definition of limit. Note that you'll need to introduce an ϵ and an $N(\epsilon)$ as part of this expansion; however, because that ϵ and $N(\epsilon)$ aren't necessarily the same as the ϵ and $N(\epsilon)$ that you introduced previously, call the new quantities ϵ_1 and $N_1(\epsilon_1)$.

2.5: We have the following ideas, all in the area of "completeness" or "closeness". Some are definitions and some are theorems.

- The limit of a sequence.
- The Bolzano-Weierstrauss Theorem. ("Every bounded sequence. . . .")
- Order completeness of $\mathbf{R}$.
- Cauchy sequences.
- Cauchy completeness.
- Density of a subset S in a metric space X.

1. For the idea assigned to you or your team, write out its definition or statement.
2. Now rotate one spot forward to a different idea and try to draw a picture of what that idea means.
3. In larger groups: Try to draw logical connections between/among:
    - The definitions of the limit of a sequence and Cauchy sequence.
    - The limit of a sequence and the Bolzano-Weierstrass Theorem: Give an example of a sequence that has no convergent subsequence.
    - Cauchy sequences and Cauchy completeness: Give an example of a space where not every Cauchy sequence has a limit.

3.1. In each of your groups, write out one of the following definitions of what it means for a function $f : X \to \mathbf{C}$ (X a nonempty subset of $\mathbf{C}$) to be continuous at $a \in X$.

- Sequential definition of continuity.
- ϵ-δ definition of continuity.

Now change places (rotate or swap) and use the other definition, the one you didn't write down, to make an outline of the proof that $f(x) = x^2$ is continuous at $x = 5$.

3.1. Give an example or explain why this is not possible. You don't need to have formulas; just draw graphs.

1. A function $f : [0, 1] \to \mathbf{R}$ that is continuous except at one point, but has neither a maximum value or a minimum value.
2. A function $f : (0, 1) \to \mathbf{R}$ that is continuous, but has neither a maximum value nor a minimum value.
3. A function $f : [0, 1] \to \mathbf{R}$ that is continuous, but has neither a maximum value nor a minimum value.
4. A function $f : \mathbf{R} \to \mathbf{R}$ that is continuous, but has neither a maximum value nor a minimum value.
5. A function $f : (0, 1) \to \mathbf{R}$ that is continuous, but is not bounded.
6. A function $f : [0, 1] \to \mathbf{R}$ that is continuous except at one point, but is not bounded.

3.1–3.2. Definition drill: Given $X \subseteq \mathbf{C}$, $a \in X$, c a limit point of X, and $f : X \to \mathbf{C}$, write out the following definitions.

- $\lim_{x \to c} f(x) = L$, sequential version.
- $\lim_{x \to c} f(x) = L$, ϵ-δ version.
- For every $a \in X$, f is continuous at a, ϵ-δ version.
- f is **uniformly** continuous on X.
- f is differentiable at a (i.e., $f'(a)$).

3.3. In teams, write one of the following definitions on the board, draw pictures to illustrate your definition, and then present your definition verbally.

1. The "Calculus II" definition of $\int_a^b v(x)\,dx$.
2. The definition of a partition of $[a, b]$.
3. The analysis definition of $\int_a^b v(x)\,dx$.

3.3. Let $v : [a, b] \to \mathbf{R}$ be bounded, and let $[x_{i-1}, x_i]$ be a subinterval of $[a, b]$. Define

$$m(v; [x_{i-1}, x_i]) = \inf \{v(x) \mid x \in [x_{i-1}, x_i]\},$$
$$M(v; [x_{i-1}, x_i]) = \sup \{v(x) \mid x \in [x_{i-1}, x_i]\}.$$

(Why do we need to use inf and sup here and not min and max?)
In teams:

1. Draw a picture of the "Calculus II" definition of $\int_a^b v(x)\,dx$, using upper and lower Riemann sums (i.e., using m and M).
2. Turn your picture into a formula-based definition of upper and lower Riemann integrals (still Calc II-style). Use:

$$\Delta x = \frac{b-a}{n} \qquad\qquad x_i = a + i\Delta x$$

3. For your Calc II definition of the integral, what happens if Δx is no longer constant, but variable? List all of the things that go wrong.
4. Write down the definition of a partition of $[a, b]$.
5. Use partitions to "fix" the Calc II definiton of the integral to becone the analysis definition of $\int_a^b v(x)\,dx$.

3.3. Let $v : [a, b] \to \mathbf{R}$ be bounded. In teams, write one of the following definitions or theorems on the board, draw pictures to illustrate your definition or theorem, and then present your definition or theorem verbally.

1. The definition of a partition of $[a, b]$.
2. The definition of an upper Riemann sum $U(v; P)$.
3. The definition of a lower Riemann sum $L(v; P)$.
4. The definition of the upper and lower Riemann integrals of v on $[a, b]$.
5. The definition of $\displaystyle\int_a^b v(x)\,dx$ and what it means for v to be integrable on $[a, b]$.
6. The sequential criterion for integrability of v on $[a, b]$.

3.4. Recall that the Sequential Criteria for Integrability Theorem says that the following are equivalent for a bounded function $v : [a, b] \to \mathbf{C}$.

- v is integrable on $[a, b]$.
- There exists a sequence of partitions P_n of $[a, b]$ such that

$$\lim_{n\to\infty} (U(v; P_n) - L(v; P_n)) = 0.$$

- For every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U(v; P_n) - L(v; P_n) < \epsilon$.

Suppose v and w are integrable on $[a, b]$. In groups, for the following problems, determine which of conditions (2) or (3) is more useful in restating the assumption and/or conclusion.

1. $v(x) + w(x)$ is integrable on $[a, b]$.
2. $\displaystyle\int_a^b (v(x) + w(x))\, dx = \int_a^b v(x)\, dx + \int_a^b w(x)\, dx.$
3. If $u : [a, b] \to \mathbf{C}$ is continuous, then u is integrable on $[a, b]$.
4. If $u : [a, b] \to \mathbf{C}$ is an increasing function, then u is integrable on $[a, b]$.

Chs. 2–3 review. For each of the following conditions, considered as either the assumption or the conclusion of a theorem, either write out what the definition of that condition tells you as an assumption, or outline what you would have to do to reach that conclusion.

1. **Assume:** $\lim_{n\to\infty} a_n = L$.
2. **Conclude:** $\lim_{n\to\infty} a_n = L$.
3. **Assume:** f is continuous at a. (ϵ-δ definition)
4. **Conclude:** f is continuous at a. (ϵ-δ definition)
5. **Assume:** f is continuous at a. (sequential definition)
6. **Conclude:** f is continuous at a. (sequential definition)
7. **Assume:** $\lim_{x\to a} f(x) = L$. (ϵ-δ definition)
8. **Conclude:** $\lim_{x\to a} f(x) = L$. (ϵ-δ definition)
9. **Assume:** $f'(a) = L$.
10. **Conclude:** $f'(a) = L$.

4.1. In teams or pairs or individually, write out the statements of the following definitions and results:

- Sequence a_n in $\mathbf{C}$
- Sequence of partial sums of a_n
- Infinite series $\displaystyle\sum_{n=k}^{\infty} a_n$
- Cauchy criterion for convergence of $\sum a_n$
- Comparison test
- $\sum a_n$ converges absolutely

Now in teams, draw a diagram illustrating the logical dependencies among these defintions and results. What depends on what?

4.2: Consider the following theorems:

Thm 1: Let $X = [0, 1]$. The sequence $f_n(x) = x^n$ converges pointwise on X to the function

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Thm 2: Let $X = \{z \in \mathbf{C} \mid |z| < 1\}$. The infinite series $\sum_{n=0}^{\infty} z^n$ converges pointwise to the function $f(z) = \dfrac{1}{1-z}$ on X.

In your team, use the definitions from 4.1 and 4.2 to outline the proof of your assigned statement, down the level of ϵ and $N(\epsilon)$.

4.2–4.3: Let X be a nonempty subset of $\mathbf{C}$, let $f_n : [0,1] \to \mathbf{C}$ be a sequence of functions, and let $f : [0,1] \to \mathbf{C}$ be a function. Write down examples of f_n that converge pointwise to f such that:

1. Each f_n is bounded, but f is not.
2. Each f_n is continuous, but f is not.
3. Each f_n is differentiable, but f is not.
4. Each f_n is differentiable and f is differentiable, but the f_n' do not converge to f'.
5. Each f_n is integrable, but f is not.
6. Each f_n is integrable and f is integrable, but $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx$.

4.3: Let X be a nonempty subset of $\mathbf{C}$, let $f_n : X \to \mathbf{C}$ be a sequence of functions, and let $f : X \to \mathbf{C}$ be a function. Write the definitions of the following two statements:

1. f_n converges to f pointwise on X.
2. f_n converges to f uniformly on X.

4.3: Use the Weierstrass M-test to prove the following statements:

1. The series $\sum_{n=0}^{\infty} x^n$ converges uniformly on $X = \left\{ x \in \mathbf{C} \,\middle|\, |x| \leq \frac{1}{2} \right\}$.
2. The series $\sum_{n=0}^{\infty} nx^n$ converges uniformly on $X = \left\{ x \in \mathbf{C} \,\middle|\, |x| \leq \frac{1}{3} \right\}$.
3. The series $\sum_{n=1}^{\infty} \frac{x^n}{n^4}$ converges uniformly on $X = \{ x \in \mathbf{C} \,|\, |x| \leq 1 \}$.

4.6: Integration bee! Integrate the following, where n is a nonzero integer. Do **NOT** use sines and cosines.

1. $\int_0^1 xe^{2n\pi ix}\, dx$
2. $\int_0^1 e^{2n\pi ix}\, dx$
3. $\int_0^1 xe^{-\pi x^2}\, dx$
4. $\int_0^1 e^{(13-5\pi i)x}\, dx$
5. $\int_0^1 x^2e^{2n\pi ix}\, dx$
6. $\int_0^1 e^{-\pi x^2}\, dx$

4.6: Unit circle pop quiz: Recall that

$$e_n(x) = e^{2\pi i n x}.$$

Express the following in terms of powers of $\pm i$, ± 1, and so on.

$$e_n(2) \qquad e_n\left(\frac{1}{2}\right) \qquad e_n\left(-\frac{1}{2}\right) \qquad e_n\left(\frac{1}{4}\right)$$

$$e_n\left(\frac{3}{4}\right) \qquad e_n(-5) \qquad e_n\left(\frac{7}{2}\right) \qquad e_n\left(-\frac{5}{4}\right)$$

5.1–5.2: Definition drill: Given $X \subseteq \mathbf{C}$, and $f : X \to \mathbf{C}$, write out the following definitions.

- $f \in C^0(X)$.
- $f \in C^r(X)$, r a positive integer.
- $f \in C^\infty(X)$.
- For $f, g \in C^0(X)$, the L^∞ distance between f and g.
- Define what it means for a function f to have domain S^1.

5.1–5.2: For $f \in \mathcal{R}([0,1])$, define

$$\begin{aligned} \|f\|_\infty &= \sup\{|f(x)| \mid x \in [0,1]\} \\ \|f\|_1 &= \int_0^1 |f(x)|\,dx \\ \|f\|_2^2 &= \int_0^1 |f(x)|^2\,dx \end{aligned}$$

If possible, find piecewise constant $f, g : [0,1] \to \mathbf{R}$ such that:

1. $\|f\|_1 < \|f\|_2^2$.
2. $\|f\|_1 > \|f\|_2^2$.
3. $\|f\|_\infty > \|g\|_\infty$ but $\|f\|_1 < \|g\|_1$.
4. $\|f\|_\infty > \|g\|_\infty$ but $\|f\|_2^2 < \|g\|_2^2$.
5. $\|f\|_1 > \|g\|_1$ but $\|f\|_2^2 < \|g\|_2^2$.

If not possible, explain why this is not possible.

6.1: Suppose that $f, g \in C^0(S^1)$, $\hat{f}(n) = 0$ for all $n \in \mathbf{Z}$, and for some $c_n \in \mathbf{C}$, $g(x) = \sum_{n \in \mathbf{Z}} c_n e_n(x)$ absolutely and uniformly on S^1.

Prove that $\int_0^1 f(x)\overline{g(x)}\, dx = 0$.

Make sure to indicate precisely where you use uniform convergence.

6.1–6.2: Definition drill: Suppose $f \in C^0(S^1)$. Define:

1. $\hat{f}(n)$
2. f_N, the N Fourier polynomial of f
3. The Fourier series of f

6.2: Compute the Fourier series of the following $f_i : S^1 \to \mathbf{C}$. Remember to do $\hat{f}(0)$ as a separate case.

$$f_1(x) = \begin{cases} -x & \text{for } -\frac{1}{2} \le x < 0, \\ 0 & \text{for } 0 \le x < \frac{1}{2}. \end{cases}$$

$$f_2(x) = \begin{cases} 0 & \text{for } -\frac{1}{2} \le x < -\frac{1}{4}, \\ x^2 & \text{for } -\frac{1}{4} \le x < \frac{1}{4}, \\ 0 & \text{for } \frac{1}{4} \le x < \frac{1}{2}. \end{cases}$$

$$f_3(x) = \begin{cases} x & \text{for } -\frac{1}{2} \le x < 0, \\ 3 & \text{for } 0 \le x < \frac{1}{2}. \end{cases}$$

$$f_4(x) = x^3 \qquad \text{for } 0 \le x < 1.$$

7.1: Let:

$$f_1(x) = \begin{cases} -x & \text{for } -\frac{1}{2} \le x < 0, \\ 0 & \text{for } 0 \le x < \frac{1}{2}. \end{cases}$$

$$f_2(x) = \begin{cases} 0 & \text{for } -\frac{1}{2} \le x < -\frac{1}{4}, \\ ix^2 & \text{for } -\frac{1}{4} \le x < \frac{1}{4}, \\ 0 & \text{for } \frac{1}{4} \le x < \frac{1}{2}. \end{cases}$$

$$f_3(x) = \begin{cases} x & \text{for } -\frac{1}{2} \le x < 0, \\ 3i & \text{for } 0 \le x < \frac{1}{2}. \end{cases}$$

$$f_4(x) = ix^3 \qquad \text{for } -\frac{1}{2} \le x < \frac{1}{2}.$$

Compute the following L^2 inner products:

$\langle f_1, f_1 \rangle$ $\quad\langle f_1, f_2 \rangle$ $\quad\langle f_1, f_3 \rangle$ $\quad\langle f_1, f_4 \rangle$

$\langle f_2, f_2 \rangle$ $\quad\langle f_2, f_3 \rangle$ $\quad\langle f_2, f_4 \rangle$ $\quad\langle f_3, f_3 \rangle$

$\langle f_3, f_4 \rangle$ $\quad\langle f_4, f_4 \rangle$

7.1: For $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbf{C}^3$, define

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Which axioms of an inner product hold for the above definition of a bracket product?

7.2: Consider the following functions on $[0,1]$. (They are not continuous functions, but they can be modified to obtain similar continuous examples, just with much more complicated formulas.)

$$f_n(x) = \begin{cases} 0 & \text{for } x = 0, \\ n & \text{for } 0 < x < \frac{1}{n}, \\ 0 & \text{for } \frac{1}{n} \leq x \leq 1. \end{cases}$$

$$g_n(x) = \begin{cases} 0 & \text{for } x = 0, \\ \sqrt{n} & \text{for } 0 < x < \frac{1}{n}, \\ 0 & \text{for } \frac{1}{n} \leq x \leq 1. \end{cases}$$

$$h_n(x) = \begin{cases} 0 & \text{for } x = 0, \\ n & \text{for } 0 < x < \frac{1}{n^2}, \\ 0 & \text{for } \frac{1}{n^2} \leq x \leq 1. \end{cases}$$

Do $f_n, g_n, h_n \to 0$ pointwise? In the L^1 metric? In the L^2 metric? Explain. (First try graphing your function.)

7.2: Suppose $f \in C^0(S^1)$. Use the continuity of the inner product (Thm. 7.2.13 and Cor. 7.2.14) to prove that if $\sum_{n \in \mathbf{Z}} c_n e_n(x)$ converges to f in the inner product metric on $C^0(S^1)$, then $\hat{f}(k) = c_k$.

7.1–7.3: Let f and g be functions (vectors) in an inner product space. Prove the following, where $\|\cdot\|$ represents the inner product norm.

1. (Pythagorean Theorem) If f and g are orthogonal, then $\|f+g\|^2 = \|f\|^2 + \|g\|^2$.
2. (Parallelogram Identity) $\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$.

7.3: For each of the following terms from generalized Fourier polynomials and series, write down the definition of the term and give a mini-lecture explaining how the term specializes to ordinary Fourier polynomials and series. (I.e., what happens when $u_n = e_n(x) = e^{2\pi inx}$ in $C^0(S^1)$?)

1. Orthogonal set
2. Orthonormal set
3. Generalized Fourier coefficient $\hat{f}(n)$
4. Projection of f onto $\mathcal{B} = \{u_1, \ldots, u_n\}$ (orthogonal set)
5. Generalized Fourier series

7.4: Write down the definition of measure zero, and then:

1. Prove that $E = \{1\}$ has measure zero.
2. Prove that $E = \{1, \ldots, n\}$ has measure zero.
3. Prove that $E = \{1, 2, \ldots\}$ (the set of positive integers) has measure zero. (Suggestion: Think geometric series.)

7.5: Definition drill: State the six Lebesgue Axioms as assigned in groups/teams/individuals.

7.5: For each X, function sequence f_n on X, and function f on X:

- Graph f_2, f_3, and f_4. Describe what happens as $n \to \infty$.
- Prove that $\lim_{n\to\infty} f_n(x) = f(x)$ pointwise for all $x \in X$ (with at most finitely many exceptions).
- Explain why $f_n(x)$ is Riemann integrable on any closed and bounded subinterval of X.
- Use our axioms for the Lebesgue integral to prove that f is measurable and Lebesgue integrable on X. Calculate $\int_X f$.

$$X = [0,1] \qquad f(x) = \frac{1}{x^{1/4}} \qquad f_n(x) = \begin{cases} 0 & \text{for } x < \frac{1}{n} \\ \frac{1}{x^{1/4}} & \text{for } x \geq \frac{1}{n} \end{cases} \tag{1}$$

$$X = [1,+\infty] \qquad f(x) = \frac{1}{x^3} \qquad f_n(x) = \begin{cases} \frac{1}{x^3} & \text{for } x \leq n \\ 0 & \text{for } x > n \end{cases} \tag{2}$$

7.6: Theorem drill: Write out the following theorems. Make the assumptions and conclusions of the theorem clear, and explain what each theorem says in the case $\mathcal{H} = L^2(S^1)$ and $u_n = e_n$.

1. Hilbert Space Absolute Convergence Theorem (Thm. 7.6.4).
2. Isomorphism Theorem for Fourier Series (Thm. 7.6.8).

7.6: We will show in Ch. 8 that $\{e_n \mid n \in \mathbf{Z}\}$ is a basis for $L^2(S^1)$; for now, assume that is true. Recall that if $f \in L^2(S^1)$ is given by $f(x) = |x|$ for $-\frac{1}{2} \leq x < \frac{1}{2}$, then

$$\hat{f}(n) = \begin{cases} \frac{1}{4} & \text{for } n = 0 \\ \dfrac{4}{(2\pi i n)^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

Use the Isomorphism Theorem for Hilbert Spaces to compute $\|f\|^2$ in two different ways. What can you conclude about $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}$?

8.2: Suppose $f, g \in C^0(S^1)$.

1. Write out the definition of $(f * g)(x)$.
2. Prove that $(f * g)(x) = (g * f)(x)$.

8.3: *Exercise in wishful thinking.* Suppose $f \in C^0(S^1)$.

1. Write out the integral definition of $\hat{f}(n)$, using t as the variable of integration.
2. Assuming that integrals can always be swapped with infinite sums, "prove" that the Fourier series $\sum_{n \in \mathbf{Z}} \hat{f}(n) e_n(x)$ is the convolution of f and some other "function" $g(x)$. What is $g(x)$, expressed as a function series?
3. Assuming all of the above wishful thinking, what would you need to be true for the Fourier series of f to converge to f?

8.4: *Overview of Fourier convergence.* For f in each of the following categories:

- State the best result that we have proven of the form "Under that condition, the Fourier series of f converges to f in the sense of (blah)."
- Identify the *type* of convergence that occurs in the result you found.
- Cite a specific theorem or theorems for the result you found.

1. $f \in L^2(S^1)$
2. $f \in C^0(S^1)$
3. $f \in C^1(S^1)$

4.7: Definition drill:

1. Define what it means for $f : \mathbf{R} \to \mathbf{C}$ to be rapidly decaying.
2. Define the Schwartz space $\mathcal{S}(\mathbf{R})$.
3. Give an example of an $f \in C^\infty(\mathbf{R})$ that is not in $\mathcal{S}(\mathbf{R})$ and an example of a rapidly decaying f that is not in $\mathcal{S}(\mathbf{R})$.

12.1: *Series vs. transform.* Fill in the following table.

	Fourier coeff $\hat{f}(n)$	Fourier transform $\hat{f}(\gamma)$
Definition		
Range/domain of f		
Range/domain of $\hat{f}$		
Condition on f for $\hat{f}$ to exist		
Inverse (transform)		
Inversion theorem		

12.2: *Limit theorems for $f * K$.* Fill in the following table.

	$f : S^1 \to \mathbf{C}$	$f : \mathbf{R} \to \mathbf{C}$
Defn of convolution		
Conditions on f, g for $f * g$ to exist		
Defn of Dirac kernel		
Example of Dirac kernel		
Limit theorem for $f * K$		