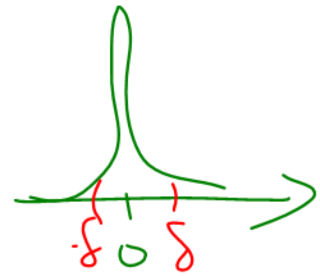
Math 131B, Fall 2022 **Name:** ______________________
Exam 3

This test consists of 8 questions on 6 pages, totalling 100 points. You are not allowed to use books, notes, or calculators. Unless otherwise stated, you may take as given anything that has been proven in class, in the homework, or in the reading.

1. (10 points) Let $K_n : [-\frac{1}{2}, \frac{1}{2}] \to \mathbf{R}$ be a sequence of **nonnegative** continuous functions. Define what it means for K_n to be a Dirac kernel on S^1. In other words, list the two key properties of K_n (besides $K_n(x) \geq 0$) that make K_n into a Dirac kernel, both having to do with the values of certain integrals of K_n. (If you can't remember the key properties precisely, for partial credit, try drawing a picture of the graph(s) of K_n.)

① $\displaystyle\int_{-1/2}^{1/2} K_n(x)\,dx = 1$



② $\forall\ 0 < \delta < \frac{1}{2}$, $\displaystyle\lim_{n\to\infty} \int_{|x|\geq\delta} K_n(x)\,dx = 0$

I.e., the portion of the integral outside a delta neighborhood of 0 goes to 0 as n -> infinity. (So portion inside delta-nbd goes to 1.)

In questions 2–4, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

2. (12 points) **TRUE/FALSE:** Suppose $h : S^1 \to \mathbf{C}$ is continuous and $K_n(x)$ is a Dirac kernel. Then it must be the case that $h * K_n$ converges uniformly to h on S^1 as $n \to \infty$.

TRUE
(Main new result of §4)
(tool)

3. (12 points) **TRUE/FALSE:** It is possible that there exist some $f \in L^2(S^1)$ and some $c_{-5}, \ldots, c_0, \ldots, c_5 \in \mathbf{C}$ such that

$$\|f - f_5\| = 2 \qquad \text{and} \qquad \left\| f - \sum_{n=-5}^{5} c_n e_n \right\| = 1,$$

should be ≥ 2

where f_5 is the Fourier polynomial of f of degree 5.

FALSE

$$\underbrace{\text{Best Approx Thm}}$$

f_5 is best poss approx to f among trig polys of deg 5, so this can't be better

$$f_5 = \sum_{n=-5}^{5} \hat{f}(n) e_n$$

4. (12 points) **TRUE/FALSE:** It is possible that there exists some $f \in C^0(S^1)$ such that $\displaystyle\sum_{n \in \mathbf{Z}} \left| \hat{f}(n) \right|^2 = 7$ and $\displaystyle\int_0^1 |f(x)|^2 \, dx = 8$.

False Isom Thm says:

$$\sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 \overset{\text{Thm}}{=} \|f\|^2 \underbrace{= \int_0^1 |f(x)|^2 dx}_{\text{defn}}$$

Can't have $7 = 8$.

5. (13 points) **PROOF QUESTION.** Let V be an inner product space and let $T : V \to V$ be a function such that T is linear, or in other words, such that for all $f, g \in V$ and $a, b \in \mathbf{C}$, we have

$$T(af + bg) = aT(f) + bT(g).$$

Also, let f_n be a sequences in V.

(a) State the **sequential** definition of what it means for $T : V \to V$ to be continuous on V.

(b) Define $\displaystyle\sum_{n=1}^{\infty} f_n$ as the limit of finite sums.

(c) Now suppose T is continuous on V and $\displaystyle\sum_{n=1}^{\infty} f_n$ converges in V. Prove that

$$T\left(\sum_{n=1}^{\infty} f_n\right) = \sum_{n=1}^{\infty} T(f_n).$$

Make sure it is clear exactly why each step of your proof is justified.

(a) If $\lim_{n\to\infty} f_n = f$, then $\lim_{n\to\infty} T(f_n) = T(f)$

I.e., $T(\lim_{n\to\infty} \; = \lim_{n\to\infty} T($

(b) $\displaystyle\sum_{n=1}^{\infty} f_n = \lim_{N\to\infty} \sum_{n=1}^{N} f_n$

(c) $\displaystyle T\left(\sum_{n=1}^{\infty} f_n\right) = T\left(\lim_{N\to\infty} \sum_{n=1}^{N} f_n\right)$ (by (b))

$\displaystyle = \lim_{N\to\infty} T\left(\sum_{n=1}^{N} f_n\right)$ (by (a), T cont.)

$\displaystyle = \lim_{N\to\infty} \sum_{n=1}^{N} T(f_n) = \sum_{n=1}^{\infty} T(f_n)$ (by (b))

T linear

6. (13 points) **PROOF QUESTION.** Let $f : S^1 \to \mathbf{C}$ be continuous, and recall that $e_n(x) = e^{2\pi i n x}$. Use the **definition** of convolution to prove that

$$(e_n * f)(x) = \hat{f}(n) e_n(x).$$

sum is outside var x

$$(e_n * f)(x) = \int_0^1 e_n(x-t) f(t)\, dt$$

$$= \int_0^1 e^{2\pi i n (x-t)} f(t)\, dt$$

$$= \int_0^1 e^{2\pi i n x} e^{-2\pi i n t} f(t)\, dt$$

$$= e^{2\pi i n x} \int_0^1 f(t) \overline{e^{2\pi i n t}}\, dt$$

$$= e_n(x) \int_0^1 f(t) \overline{e_n(t)}\, dt$$

$$= e_n(x) \hat{f}(n)$$

7. (14 points) **PROOF QUESTION.** Let $\{e_n \mid n \in \mathbf{Z}\}$ be the usual basis for $L^2(S^1)$ and suppose $a_n \in \mathbf{C}$.

(a) State the Hilbert Space Absolute Convergence Theorem for $\sum_{n\in\mathbf{Z}} a_n e_n$.

(b) Now suppose that for all $n \in \mathbf{Z}$,

$$|a_n| \leq \frac{7}{|n|+2}.$$

Prove that $\sum_{n\in\mathbf{Z}} a_n e_n$ converges (in the inner product metric) to some $f \in L^2(S^1)$.

(a) $\sum_{n\in\mathbb{Z}} a_n e_n$ convs in $L^2(S^1)$

$\iff \sum_{n\in\mathbb{Z}} |a_n|^2$ convs in $\mathbb{R}$

(b) $|a_n|^2 \leq \left(\frac{7}{|n|+2}\right)^2 \leq \left(\frac{7}{n^2}\right) (n\neq 0)$

$\sum_{n\neq 0} \frac{7}{n^2}$ convs (p-series, $p=2>1$)

so $\sum_{n\in\mathbb{Z}} |a_n|^2$ convs by comparison.

$\Rightarrow \sum_{n\in\mathbb{Z}} a_n e_n$ convs in $L^2(S^1)$ (HSACT)

8. (14 points) **PROOF QUESTION.** Let $k : S^1 \to \mathbf{R}$ be a nonnegative continuous function such that

$$\int_0^1 k(t)\,dt = 1.$$

assume in (b)

(a) Define what it means for $f : S^1 \to \mathbf{C}$ to be uniformly continuous. (For partial credit, don't worry about the "uniformly" part.)

(b) Prove that for every $\epsilon > 0$, there exists some $\delta > 0$ such that for any x such that $|x| < \delta$, we have that

$$\left| \int_0^1 (f(t+x) - f(t))k(t)\,dt \right| \le \epsilon.$$

(a) $\forall \epsilon_1 > 0 \ \exists \delta_1(\epsilon_1) > 0$ s.t. $\forall x, y \in S^1$, if $|x-y| < \delta_1(\epsilon_1)$, then $|f(x) - f(y)| < \epsilon_1$.

(b) $\forall \epsilon > 0$ $\quad \epsilon_1 = \epsilon/2$

Let $\delta = \delta_1\left(\frac{\epsilon}{2}\right) \Rightarrow$ $\boxed{\text{If } |y-z| < \delta \text{ then } |f(y) - f(z)| < \frac{\epsilon}{2}}$

$\forall$ $|x| < \delta$

$$\left|\int_0^1 (f(t+x) - f(t))k(t)\,dt\right|$$

$$\le \int_0^1 |f(t+x) - f(t)|\,|k(t)|\,dt$$

$$< \int_0^1 \frac{\epsilon}{2} |k(t)|\,dt = \frac{\epsilon}{2}\int_0^1 k(t)\,dt = \frac{\epsilon}{2} < \epsilon$$

(c) $\left|\int_0^1 (f(t+x) - f(t))k(t)\,dt\right| < \epsilon$

This is the red box, but with (t+x) and t instead of x and y

$|f(t+x) - f(t)| < \frac{\epsilon}{2}$ b/c $|(t+x) - t| = |x| < \delta$