Sample Exam 3
Math 131A, Spring 2024

1. (20 points) The goal of this problem is to define the Riemann integral.

Let $f : [a, b] \to \mathbf{R}$ be a bounded real-valued function. Recall that a *partition* P of $[a, b]$ is a finite subset $\{x_0, \dots, x_n\} \subset [a, b]$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, and let $\mathcal{P}$ be the set of all partitions of $[a, b]$.

(a) (Optional) Draw a picture of a **lower** Riemann sum. You can get full credit for this problem without drawing this picture, but you may find it helpful, and you can get partial credit just based on your picture.

(b) For a fixed partition $P \in \mathcal{P}$, define $m(f; P, i)$.

(c) For a fixed partition $P \in \mathcal{P}$, define the **lower** Riemann sum $L(f; P)$.

(d) Define the **lower** Riemann integral $\underline{\int_a^b} f(x)\, dx$

(e) Define what it means for f to be integrable on $[a, b]$, and define $\int_a^b f(x)\, dx$.

In questions 2–5, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

2. (11 points) **TRUE/FALSE:** Let $h : \mathbf{R} \to \mathbf{R}$ be a function such that

$$\lim_{x \to 4} \frac{h(x) - h(4)}{x - 4} = 7.$$

Then it must be the case that h is continuous at $x = 4$.

3. (11 points) **TRUE/FALSE:** Let $g : [-3, 2] \to \mathbf{R}$ be a continuous function. Then it is possible that $\{g(x) \mid -3 \leq x \leq 2\}$ is precisely equal to the open interval $(5, 13)$.

4. (11 points) **TRUE/FALSE:** Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Then $\lim_{x \to c} f(x)$ exists for every $c \in \mathbf{R}$.

5. (11 points) **TRUE/FALSE:** Let $k : \mathbf{R} \to \mathbf{R}$ be a function that is continuous at $x = 0$. Then it must be the case that k is differentiable at $x = 0$.

6. (12 points) **PROOF QUESTION.** Let $f : [0, +\infty) \to \mathbf{R}$ be a function that is differentiable on $(0, +\infty)$ and continuous on $[0, +\infty)$ such that

$$f'(x) = 2 + 3\sin\left(\frac{1}{x}\right) \qquad \text{for } x > 0,$$
$$f(0) = 7.$$

Use the Mean Value Theorem to prove that for any $x > 0$, we have that $7 - x \leq f(x) \leq 7 + 5x$.

7. (12 points) **PROOF QUESTION.** Let $g : \mathbf{R} \to \mathbf{R}$ be defined by

$$g(x) = \begin{cases} x^2 & \text{if } x \notin \mathbf{Q}, \\ 0 & \text{if } x \in \mathbf{Q}. \end{cases}$$

Prove that $g'(0) = 0$.

8. (12 points) **PROOF QUESTION.** Let $A = \mathbf{R} \setminus \{0\}$ (i.e., A is all real numbers except 0), and let $h : A \to \mathbf{R}$ be defined by

$$h(x) = \sqrt[3]{x} \cos\left(\frac{1}{x}\right).$$

Use the definition of the limit (ϵ-δ or sequential) to prove that

$$\lim_{x \to 0} h(x) = 0.$$

(Using the Squeeze Lemma will only get you partial credit, not full credit.)