Sample Exam 2
Math 131A, Spring 2024

1. (10 points) Determine if the series $\sum_{n=0}^{\infty} \frac{12^n(3+2^{-n})}{n!}$ converges or diverges, and prove your answer.

2. (10 points) Determine if the series $\sum_{n=1}^{\infty} \frac{(n^3-7n^2)n^{1/n}}{n^4+3n}$ converges or diverges, and prove your answer.

3. (11 points) State the Bolzano-Weierstrass Theorem.

In questions 4–6, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

4. (11 points) **TRUE/FALSE:** Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then f is continuous at 0.

5. (11 points) **TRUE/FALSE:** Let s_n be a sequence such that $s_n \geq s_{n+1} \geq -8$ for all $n \in \mathbf{N}$. Then it is possible that s_n does not converge.

6. (11 points) **TRUE/FALSE:** Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function such that $f(r) = -3$ for all rational numbers $r \in \mathbf{Q}$. Then it must be the case that $f(\pi) = -3$.

7. (12 points) **PROOF QUESTION.** Define $g : \mathbf{R} \to \mathbf{R}$ by

$$g(x) = \begin{cases} |x| & \text{if } x \in \mathbf{Q}; \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Prove that g is continuous at 0.

8. (12 points) **PROOF QUESTION.** Let s_n be a bounded sequence such that $s_n \leq 4$ for all $n \in \mathbf{N}$, and let

$$S_k = \{s_n \mid n \geq k\}, \qquad b_k = \sup S_k.$$

(a) What is the standard name of $\lim_{k\to\infty} b_k$?

(b) Prove that $b_k \leq 4$ for all $k \in \mathbf{N}$.

(c) Prove that $\limsup_{n\to\infty} s_n \leq 4$.

9. (12 points) **PROOF QUESTION.** Suppose that a_n and b_n are sequences of positive real numbers, and suppose that there exists some $M > 0$ such that

$$\frac{a_n}{b_n} \leq M$$

for all $n \in \mathbf{N}$. Exactly one of the following statements is always true.

- If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.
- If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

CIRCLE the statement that is always true, and prove that statement.