Math 131A, problem set 11
Outline due: Fri May 11
Completed version due: Mon May 14
Last revision due: TBA

Problems to be done but not turned in: 25.1, 25.3, 25.5, 25.7, 25.9, 25.11, 25.13, 25.15, 26.1, 26.3, 26.5, 26.7.

Problems to be turned in: All numbers refer to exercises in Ross.

1. Define $g : I \to \mathbf{R}$ by $g(x) = \sum_{n=1}^{\infty} \frac{3^n x^n}{n^{3/2}}$, where I is the interval of convergence of the series.
   (a) Compute the radius of convergence of g, and compute I (i.e., what happens at the boundary?), with proof.
   (b) Prove that g is continuous on I. (Suggestion: Weierstrass M-test.)
2. Ex. 25.10(a,b).
3. Ex. 25.12.
4. Define $f : (-R, R) \to \mathbf{R}$ by $f(x) = \sum_{n=1}^{\infty} n^2 x^n$, where R is the interval of convergence of the series.
   (a) Compute R, with proof.
   (b) Find a closed (non-series) formula for $f(x)$, with proof.
   (c) Find the exact value of $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{7^n}$, with proof.
5. For the rest of this problem set, let

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

   You may use the fact (proven in PS10) that the radii of convergence of both $S(x)$ and $C(x)$ are each equal to ∞, and so $S(x)$ and $C(x)$ are well-defined functions on $\mathbf{R}$. You are **not** allowed to assume that S and C are sine and cosine, as our goal is to rigorously construct the sine and cosine functions.

   (a) Prove that $S(0) = 0$, $C(0) = 1$, $S'(x) = C(x)$, and $C'(x) = -S(x)$. Justify convergence carefully.
   (b) Prove that for $u, v : \mathbf{R} \to \mathbf{R}$, if $u'(x) = v(x)$, $v'(x) = -u(x)$, and $f(x) = (u(x))^2 + (v(x))^2$, then $f'(x) = 0$. What conclusion can you draw about f?
   (c) Prove that $S(x)^2 + C(x)^2 = 1$ for all $x \in \mathbf{R}$.

   (Cont. on next page.)

(d) For fixed $a \in \mathbf{R}$, use part (b) and the functions

$$S_a(x) = S(x+a) - S(x)C(a) - C(x)S(a),$$
$$C_a(x) = C(x+a) - C(x)C(a) + S(x)S(a),$$

to prove that, for all $x \in \mathbf{R}$,

$$S(x+a) = S(x)C(a) + C(x)S(a),$$
$$C(x+a) = C(x)C(a) - S(x)S(a).$$

6. The goal of this problem is to prove that there exists some $x > 0$ such that $C(x) = 0$. Proceeding by contradiction, assume for the entirety of this problem that $C(x) > 0$ for all $x > 0$.

(a) Let $m = S(1)$. Prove that $m = S(1) > 0$.

(b) Prove that $C'(x) < -m$ for all $x > 1$.

(c) Prove that for $x > 1$, $C(x) - C(1) < -m(x-1)$. Obtain a contradiction with the assumption $C(x) > 0$ for all $x > 0$.

7. **TO BE DONE BUT NOT TURNED IN.** The goal of this problem is to prove that S and C are periodic.

(a) Let $T = \{x \in \mathbf{R} \mid x > 0 \text{ and } C(x) = 0\}$ and $x_0 = \inf T$. Prove that $C(x_0) = 0$. (Suggestion: Use the Arbtrarily Close Criterion to construct a sequence in T whose limit is x_0. Why is C continuous?)

(b) Prove that for $0 < x < x_0$, $S(x)$ is strictly increasing, and so $S(x_0) = 1$.

(c) Now define $\pi = 2x_0$, which means that by definition, $x_0 = \dfrac{\pi}{2}$. Prove that for all $x \in \mathbf{R}$, $S(x + 2\pi) = S(x)$ and $C(x + 2\pi) = C(x)$. (Suggestion: Use both the previous parts of this problem and also problem 5.)