Math 131A, problem set 08
Outline due: Wed Apr 11
Completed version due: Mon Apr 13
Last revision due: Mon May 14

Problems to be done but not turned in: 20.13, 20.15, 20.17, 20.19; 28.1, 28.3, 28.5, 28.7, 28.9, 28.11, 28.13, 28.15.

Problems to be turned in: All numbers refer to exercises in Ross.

1. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by
$$f(x) = \begin{cases} 1 & \text{for } x \in \mathbf{Q}, \\ 0 & \text{for } x \notin \mathbf{Q}. \end{cases}$$
Prove that for any $c \in \mathbf{R}$, $\lim_{x \to c} f(x)$ does not exist.

2. Let $g : \mathbf{R} \setminus \{3\} \to \mathbf{R}$ be defined by $g(x) = \dfrac{1}{\sqrt[3]{x-3}}$.
   (a) Prove that $\lim_{x \to 3^+} g(x) = +\infty$.
   (b) Prove that $\lim_{x \to -\infty} g(x) = 0$.

3. Ex. 28.2(d).

4. Prove the *Local Linearity Lemma:*

   Local Linearity Lemma. Let I be an interval and let $f : I \to \mathbf{R}$ be a function. For $a \in I$, the following are equivalent:
   - f is differentiable at a and there exists some $m \in \mathbf{R}$ such that $m = f'(a)$.
   - There exists $m \in \mathbf{R}$ and a function $\epsilon(x)$ (the *relative error* in the local linear approximation to f at a) such that
   $$f(x) - f(a) = (x-a)(m + \epsilon(x))$$
   and $\lim_{x \to a} \epsilon(x) = 0 = \epsilon(a)$ (i.e., $\epsilon(x)$ is continuous at a and $\epsilon(a) = 0$).

   (Note that as a consequence, if either condition is true, and therefore, both conditions are true, then $m = f'(a)$.)

5. Let $h : \mathbf{R} \to \mathbf{R}$ be defined by
$$h(x) = \begin{cases} 2x^2 \sin\left(\dfrac{1}{x}\right) + x & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$
   (a) Prove that $h'(0) = 1$.
   (b) Use the laws of differentiation to calculate $h'(a)$ for $a \neq 0$.
   (c) Prove that for any $\epsilon > 0$, there exists some $a \in \mathbf{R}$ such that $0 < a < \epsilon$ and $h'(a) < 0$.

(Cont. on next page.)

6. Let $r \in \mathbf{R}$ be fixed, and let $k_r : \mathbf{R} \to \mathbf{R}$ be defined by

$$k_r(x) = \begin{cases} |x|^r & \text{for } x \in \mathbf{Q} \text{ and } x \neq 0, \\ 0 & \text{for } x \notin \mathbf{Q} \text{ or } x = 0. \end{cases}$$

Determine and prove the best possible theorem of the following form. (That is, prove all three statements below, with the best possible values of the ??s.)

Theorem. We have that:

- For $r >$??, $k_r(x)$ is differentiable at 0.
- For ?? $< r \leq$??, $k_r(x)$ is continuous but not differentiable at 0.
- For $r \leq$??, $k_r(x)$ is not continuous at 0.