

Math 131A, problem set 06
Outline due: Wed Mar 14
Completed version due: Wed Mar 21
Last revision due: Wed Apr 25

Problems to be done but not turned in: 15.1, 15.3, 15.5, 15.7; 17.1, 17.3, 17.5, 17.7, 17.9, 17.11, 17.13, 17.15.

Problems to be turned in: All numbers refer to exercises in Ross.

1. Ex. 15.4(b,d).
2. Ex. 15.6.
3. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x) = \sqrt[3]{x}$. Use the definition of continuity to prove that f is continuous at 0.
4. For a nonempty $A \subseteq \mathbf{R}$ and $K > 0$, we say that a function $g : A \rightarrow \mathbf{R}$ is K -Lipschitz if for any $x, y \in A$, we have

$$|g(x) - g(y)| \leq K |x - y|.$$

Let $A \subseteq \mathbf{R}$ be nonempty and $K > 0$, and suppose that $g : A \rightarrow \mathbf{R}$ is K -Lipschitz. Prove that g is continuous on A (i.e., at every $a \in A$).

5. Ex. 17.10(b,c).
6. Let $h : \mathbf{R} \rightarrow \mathbf{R}$ be defined by

$$h(x) = \begin{cases} \sqrt{x} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove or disprove: h is continuous at 0.

7. Define $f : [0, 1] \rightarrow \mathbf{R}$ by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = p/q, p, q \in \mathbf{Z}, q > 0, \gcd(p, q) = 1; \\ 0 & \text{if } x \text{ irrational.} \end{cases}$$

- (a) Prove that if x is rational, then f is discontinuous at x . (See the hint for Ex. 17.13; note that $f(0) = 1$.)
- (b) Prove that if x is irrational, then f is continuous at x . (Suggestion: For which x is $f(x) \geq 1/n$?)