Math 131A, problem set 04
Outline due: Wed Feb 28
Completed version due: Mon Mar 05
Last revision due: Wed Mar 21

Additional definition: For $k \in \mathbf{N}$, the *k-tail* of a sequence $(s_n)_{n \in \mathbf{N}}$ is the sequence $(s_n)_{n \geq k}$, i.e., the sequence starting at k instead of 1.

Problems to be done but not turned in: 10.1, 10.3, 10.5, 10.7, 10.9, 10.11, 11.1, 11.3, 11.5, 11.7, 11.9, 11.11; and:

- For $k \in \mathbf{N}$, prove that a sequence (s_n) converges if and only the k-tail of (s_n) converges.
- This problem goes through some of the details of the definitions of lim inf and lim sup. (The answers are essentially in Sect. 10.) Let L, M be real numbers, and let s_n be a sequence such that $L \leq s_n \leq M$ for all n. Define:

$$S_k = \{s_n \mid n \geq k\}$$
$$M_k = \sup S_k,$$
$$L_k = \inf S_k.$$

  - For $k \in \mathbf{N}$, is $S_k \subseteq S_{k+1}$ or $S_{k+1} \subseteq S_k$? Prove your answer.
  - For $k \in \mathbf{N}$, is $M_k \geq M_{k+1}$ or $M_k \geq M_{k+1}$? Prove your answer. State (without proof) the analogous result for L_k.
  - Prove that for $k \in \mathbf{N}$, $L_k \leq M$. State (without proof) the analogous fact for M_k.
  - Prove that $\lim_{k\to\infty} M_k$ and $\lim_{k\to\infty} L_k$ both exist. Explain why this means that $\limsup s_n$ and $\liminf s_n$ are well-defined.

Problems to be turned in: All numbers refer to exercises in Ross.

1. Ex. 9.14, cases $|a| \leq 1$ and $a > 1$ only. (I.e., you do not have to turn in the case where $a < -1$.) You may also use the fact, to be proven later, that if $p > 0$ and (b_n) is a convergent sequence such that $b_n \geq 0$, then $\lim(b_n^p) = (\lim b_n)^p$.
2. Ex. 10.2.
3. Let p_n be the nth prime number (so $p_1 = 2$, $p_4 = 7$, and so on). Define a sequence s_n inductively by

$$s_1 = 13, \qquad s_{n+1} = \left(1 - \frac{1}{p_n}\right) s_n.$$

   Prove that $\lim s_n$ exists.
4. Ex. 11.2(a,c,d,e).

(cont. on next page)

5. Note that for closed intervals $[a, b]$ and $[c, d]$ in $\mathbf{R}$, we have that $[a, b] \supseteq [c, d]$ exactly when $a \leq c \leq d \leq b$.

   Suppose that

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots \supseteq [a_n, b_n] \supseteq \cdots$$

   is an infinite sequence of nested closed bounded intervals.

   (a) Prove that a_n is an increasing sequence, and that for any $k \in \mathbf{N}$, b_k is an upper bound for the sequence a_n.

   (b) Prove that $a = \lim a_n$ and $b = \lim b_n$ exist.

   (c) Prove that $a = \sup a_n \leq \inf b_n = b$. (Suggestion: First prove that for any $k \in \mathbf{N}$, $\sup a_n \leq b_k$.)

6. This problem gives an alternate proof of Bolzano-Weierstrass, using problem 5.

   Suppose s_n is a sequence in the closed interval $[a_1, b_1]$. We inductively choose a sequence of nested closed intervals $[a_k, b_k]$ of length $\left(\dfrac{b_1 - a_1}{2^{k-1}}\right)$, and a subsequence $t_k = s_{n_k}$ of s_n such that $t_k \in [a_k, b_k]$, as follows:

   - Let $t_1 = s_1$; $t_1 \in [a_1, b_1]$ by hypothesis.
   - By induction, we may assume, for a particular $k \geq 1$, that $[a_k, b_k]$ contains infinitely many terms from the sequence s_n occurring after the subsequence term t_k. It follows that either the first half of $[a_k, b_k]$ or the second half of $[a_k, b_k]$ contain infinitely many terms from s_n occurring after t_k, so choose one of the halves of $[a_k, b_k]$ containing infinitely many terms from s_n occurring after t_k, and let t_{k+1} be one of those terms.

   In any case, you may safely assume that there exist intervals $[a_k, b_k]$ and a subsequence $t_k = s_{n_k}$ of s_n as described above.

   (a) Let $a = \sup a_k$ and $b = \inf b_k$. Prove that $a = b$. (Suggestion: Use problem 5 to prove that for any k, $0 \leq b - a \leq \dfrac{b_1 - a_1}{2^{k-1}}$. Then use the Squeeze Lemma.)

   (b) Prove that t_k converges.

7. Let L, M be real numbers, and let s_n and t_n be sequences such that $L \leq s_n \leq M$ and $L \leq t_n \leq M$ for all n. Define:

$$\begin{aligned} S_k &= \{s_n \mid n \geq k\}, & M_{S,k} &= \sup S_k, \\ T_k &= \{t_n \mid n \geq k\}, & M_{T,k} &= \sup T_k, \\ U_k &= \{s_n + t_n \mid n \geq k\}, & M_{U,k} &= \sup U_k. \end{aligned}$$

   (a) What is the standard name of $\lim_{k \to \infty} M_{S,k}$?

   (b) Prove that for all k, $M_{U,k} \leq M_{S,k} + M_{T,k}$. (Suggestion: Use the sup trick.)

   (c) Prove that $\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$.