Notes on the Riemann integral
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1 The Riemann integral: Definition

In calculus, for nonnegative $f(x)$, the Riemann integral is often described as follows: Divide $[a,b]$ into n equal subintervals of size $\Delta x = \dfrac{b-a}{n}$, choose a rectangle on each subinterval whose height is some value of $f(x)$ on that subinterval, define the corresponding *Riemann sum* to be the sum of the areas of those rectangles, and define the *Riemann integral* $\displaystyle\int_a^b f(x)\,dx$ to be the limit of Riemann sums as $n \to \infty$, if the limit exists. For a continuous $f(x)$, this might involve choosing "upper" or "lower" Riemann sums, as shown in Figure 1.1. In any case, after writing this process out formally, it turns out that we do not really need $f(x)$ to be nonnegative, and in fact, for any continuous $f(x)$, the limit exists and does not depend on our choices of the points x_i^* in each subinterval at which we calculate the rectangle heights $f(x_i^*)$.

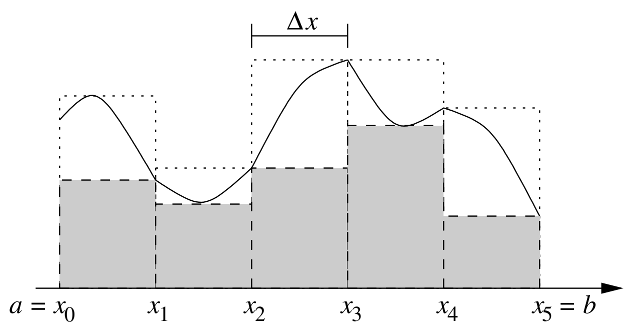

Figure 1.1: The "calculus" version of Riemann sums

Now, as we shall see (Remark 1.3), this simplified definition of the Riemann integral has technical shortcomings; the main issue is that it will be helpful to consider *unequal* subdivisions of $[a,b]$ (Definition 1.1), even at the cost of requiring additional technical complexity in our definitions (see Definition 1.6). Nevertheless, in the end, the morally correct definition is not that different from the "equal subdivisions" definition, and in truth, the reader who nods and smiles and pretends to go along with the fancy version, but is secretly thinking about the calculus version, should mostly be fine.

For the rest of this section, let $[a,b]$ be a nonempty finite closed interval in $\mathbf{R}$, and let $f : [a,b] \to \mathbf{R}$ be bounded. As promised, we begin by defining terminology describing unequal subdivisions of $[a,b]$.

Definition 1.1. A *partition* P of $[a,b]$ is a finite subset $\{x_0, \ldots, x_n\} \subset [a,b]$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. We call $[x_{i-1}, x_i]$ the *ith subinterval* of P, and when P is understood, we use the abbreviation $(\Delta x)_i = x_i - x_{i-1}$. See Figure 1.2.

Suppose P and Q are partitions of $[a,b]$. If $P \subseteq Q$, we say that Q is a *refinement* of P, as each subinterval of P is the union of of one or more subintervals of Q. Similarly, we use $P \cup Q$ to denote the partition of $[a, b]$ obtained from the points of $P \cup Q$, written in ascending order, and we call $P \cup Q$ the *common refinement* of P and Q.

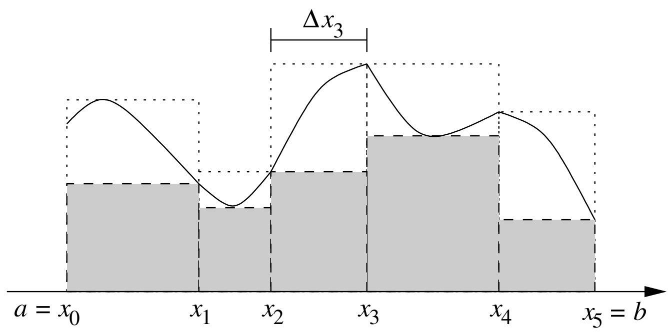

Figure 1.2: Partitions and upper and lower Riemann sums

Example 1.2. Let n be a positive integer. The *nth standard partition of* $[a, b]$ is defined by letting $\Delta x = \dfrac{b-a}{n}$, and defining $x_i = a + i\Delta x$ for $0 \leq i \leq n$, so that $x_0 = a$, $x_n = b$, and the other x_i are evenly spaced. See Figure 1.1.

Remark 1.3. Note that if P is a partition of $[a, b]$ and Q is a partition of $[b, c]$, then $P \cup Q$ (in ascending order) is a partition of $[a, c]$. The fact that this fails for standard partitions is one of the main reasons why partitions are useful for defining the Riemann integral.

Next, we define upper and lower Riemann sums corresponding to a partition, taking our variably-sized $(\Delta x)_i$ into account.

Definition 1.4. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. Since we continue to assume that $f(x)$ is bounded, we can define

$$\begin{aligned} M(f; P, i) &= \sup \{f(x) \mid x \in [x_{i-1}, x_i]\}, \\ m(f; P, i) &= \inf \{f(x) \mid x \in [x_{i-1}, x_i]\}. \end{aligned} \tag{1.1}$$

We define the *upper Riemann sum* $U(f; P)$ to be

$$U(f; P) = \sum_{i=1}^{n} M(f; P, i)(\Delta x)_i, \tag{1.2}$$

and we define the *lower Riemann sum* $L(f; P)$ to be

$$L(f; P) = \sum_{i=1}^{n} m(f; P, i)(\Delta x)_i. \tag{1.3}$$

In other words, we define Riemann sums as in calculus, except that we use sups and infs instead of values of $f(x)$, and $(\Delta x)_i$ is variable, not constant. Again, see Figure 1.2.

Example 1.5. Let P be the nth standard partition of $[a, b]$. If $f(x)$ is continuous, then by the Extreme Value Theorem, $M(f; P, i)$ and $m(f; P, i)$ are both actual (maximum and minimum) values of $f(x)$ on the ith subinterval of P, so $U(f; P)$ and $L(f; P)$ are Riemann sums in the calculus sense. Again, see Figure 1.1.

The one other technical wrinkle we need to add to the calculus version of the integral comes from the fact that partitions are not indexed by some integer variable n. (In fact, there are uncountably many partitions of $[a, b]$.) Therefore, we need to replace the former $\lim_{n\to\infty}$ with sups and infs, as follows.

Definition 1.6. Let $\mathcal{P}$ be the set of all partitions of $[a, b]$. We define the *upper Riemann integral* and *lower Riemann integral* of f on $[a, b]$ to be

$$\overline{\int_a^b} f(x)\, dx = \inf \{U(f; P) \mid P \in \mathcal{P}\}, \tag{1.4}$$

$$\underline{\int_a^b} f(x)\, dx = \sup \{L(f; P) \mid P \in \mathcal{P}\}, \tag{1.5}$$

respectively. Note that (at least in pictures, see Figure 1.2) each $U(f; P)$ is an overestimate of the area under $y = f(x)$ and each $L(f; P)$ is an underestimate of the area under $y = f(x)$, so we may think of (1.4) and (1.5) as best possible upper and lower estimates to the area under the curve, respectively.

We may now finally define the Riemann integral of a function. The careful reader should note that we hereby explicitly require integrable functions to be bounded.

Definition 1.7. To say that f is *integrable* on $[a, b]$ means that f is bounded on $[a, b]$ and the upper and lower integrals of f on $[a, b]$ are equal. If f is integrable, we define the *Riemann integral of f on* $[a, b]$ to be

$$\int_a^b f(x)\, dx = \overline{\int_a^b} f(x)\, dx = \underline{\int_a^b} f(x)\, dx. \tag{1.6}$$

Having laboriously defined the Riemann integral, we now come to a equally difficult task: Actually proving that any useful examples of functions are integrable. Most of the rest of this section is therefore devoted to obtaining a technical tool used for that purpose, the *Sequential Criterion for Integrability* (Lemma 1.10). First, we have some preliminary results on upper and lower sums and integrals.

Lemma 1.8. *Let $f : [a, b] \to \mathbf{R}$ be bounded and let $Q \subseteq P$ be partitions of $[a, b]$. Then*

$$L(f; Q) \le L(f; P) \le U(f; P) \le U(f; Q). \tag{1.7}$$

Note that (1.7) actually makes two claims: The lower sum corresponding to a given partition is never greater than the corresponding upper sum, and refining a partition Q to a partition P produces "closer" lower and upper sums.

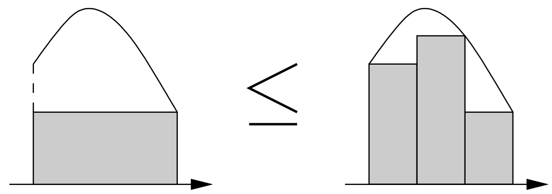

Figure 1.3: Subdividing a subinterval into k pieces ($k = 3$)

Proof. The middle inequality in (1.7) is proven in Problem 1.1. As for the other inequalities in (1.7), we only prove the first one, as the last follows by an analogous argument. In fact, since P is obtained by subdividing each subinterval of Q into one or more pieces, it will suffice to consider what happens when we subdivide a single subinterval.

For notational simplicity, suppose the first subinterval $[x_0, x_1]$ of Q is divided into k pieces in P; more precisely, suppose the first k subintervals of P are $[t_0, t_1], \ldots, [t_{k-1}, t_k]$, where $x_0 = t_0$ and $x_1 = t_k$. It follows that in the respective lower Riemann sums, we end up replacing the term $m(f; Q, 1)(\Delta x)_1$ with the sum $\displaystyle\sum_{j=1}^{k} m(f; P, j)(\Delta t)_j$.

Now, since $[t_{j-1}, t_j] \subseteq [x_0, x_1]$ for $1 \le j \le k$, we see that

$$\begin{aligned} m(f; Q, 1) &= \inf \{f(x) \mid x \in [x_0, x_1]\} \\ &\le \inf \{f(x) \mid x \in [t_{j-1}, t_j]\} = m(f; P, j). \end{aligned} \tag{1.8}$$

In other words, the inf of each smaller subinterval is at least as large as the inf of the original subinterval; compare the two sides of Figure 1.3. It follows that, as shown in Figure 1.3,

$$\begin{aligned} \sum_{j=1}^{k} m(f; P, j)(\Delta t)_j &\ge \sum_{j=1}^{k} m(f; Q, 1)(\Delta t)_j \\ &= m(f; Q, 1) \sum_{j=1}^{k} (\Delta t)_j = m(f; Q, 1)(\Delta x)_1. \end{aligned} \tag{1.9}$$

Summing the analogous inequalities over the subintervals of Q, we see that $L(f; Q) \le L(f; P)$, and the lemma follows. $\square$

One point glossed over in Definition 1.6 is the question of whether the upper integral, defined to be the inf of some set, is a real number or merely $-\infty$, and similarly for the lower integral. The following result clarifies this point and the relation of the upper and lower integrals to upper and lower sums.

Theorem 1.9. *Let $f : [a, b] \to \mathbf{R}$ be bounded. Then $\underline{\int_a^b} f(x)\,dx$ and $\overline{\int_a^b} f(x)\,dx$ are both real numbers (and not $\pm\infty$), and for any partition P of $[a, b]$, we have*

$$L(f; P) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(f; P). \tag{1.10}$$

Proof. Let $\mathcal{P}$ be the set of all partitions of $[a,b]$, let $\mathcal{L} = \{L(f;P) \mid P \in \mathcal{P}\}$, and let $\mathcal{U} = \{U(f;P) \mid P \in \mathcal{P}\}$. Problem 1.2 shows that $\mathcal{L}$ is bounded above and $\mathcal{U}$ is bounded below, which means that $\underline{\int_a^b} f(x)\,dx$ and $\overline{\int_a^b} f(x)\,dx$ are real. Problem 1.2 also proves (1.10). $\square$

Lemma 1.10 (Sequential Criteria for Integrability)**.** *Let $f : [a,b] \to \mathbf{R}$ be bounded. Then the following are equivalent.*

1. *f is integrable on $[a,b]$.*
2. *There exists a sequence of partitions P_n such that $\lim_{n\to\infty} (U(f;P_n) - L(f;P_n)) = 0$.*
3. *For any $\epsilon > 0$, there exists a partition P such that $U(f;P) - L(f;P) < \epsilon$.*

Furthermore, if condition (2) holds, then

$$\lim_{n\to\infty} L(f;P_n) = \int_a^b f(x)\,dx = \lim_{n\to\infty} U(f;P_n). \tag{1.11}$$

Note that as the name implies, Lemma 1.10 once again reduces integrability and computing the integral to the problem of computing the limit of some sequence of Riemann sums.

Proof. Let $\mathcal{P}$ be the set of all partitions of $[a,b]$, let $\mathcal{L} = \{L(f;P) \mid P \in \mathcal{P}\}$, and let $\mathcal{U} = \{U(f;P) \mid P \in \mathcal{P}\}$.

(1)$\Rightarrow$(2): Let n be a positive integer. By the Arbitrarily Close Criterion, there exists some $Q_n \in \mathcal{P}$ such that

$$\int_a^b f(x)\,dx - L(f;Q_n) = \underline{\int_a^b} f(x)\,dx - L(f;Q_n) < \frac{1}{2n}, \tag{1.12}$$

and there exists some $Q'_n \in \mathcal{P}$ such that

$$U(f;Q'_n) - \int_a^b f(x)\,dx < \frac{1}{2n}. \tag{1.13}$$

By Lemma 1.8, taking the common refinement $P_n = Q_n \cup Q'_n$ pushes the quantities $U(f;Q'_n)$ and $L(f;Q_n)$ closer together, and so

$$\begin{aligned} U(f;P_n) - L(f;P_n) &\le U(f;Q'_n) - L(f;Q_n) \\ &= U(f;Q'_n) - \int_a^b f(x)\,dx + \int_a^b f(x)\,dx - L(f;Q_n) < \frac{1}{n}. \end{aligned} \tag{1.14}$$

Condition (2) then follows by the Squeeze Lemma.

(2)$\Rightarrow$(3): Given $\epsilon > 0$, by the definition of limit, we may let $P = P_n$ for any sufficiently large n.

(3)$\Rightarrow$(1): Let $\Delta = \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \ge 0$ (Theorem 1.9). If condition (3) holds, by (1.10), $\Delta < \epsilon$ for any $\epsilon > 0$, and so $\Delta = 0$.

Finally, suppose condition (2) holds. In that case, by (1.10), we have that for any n,

$$0 \leq \underline{\int_a^b} f(x)\,dx - L(f;P_n) \leq U(f;P_n) - L(f;P_n). \tag{1.15}$$

So by the Squeeze Lemma,

$$\lim_{n\to\infty} L(f;P_n) = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx, \tag{1.16}$$

and the theorem follows. ☐

We conclude with a single, admittedly silly example. Nevertheless, the reader should carry out this computation, not just because this result actually becomes useful later, but also as an exercise in understanding the many layers of definitions established in this section.

Lemma 1.11. *Constant functions are integrable, and for $c \in \mathbf{R}$, $\displaystyle\int_a^b c\,dx = c(b-a)$.*

Proof. See Problem 1.3. ☐

See also Problem 1.4 for an example of a nonintegrable function.

Problems

1.1. (*Proves Lemma 1.8*) Let $f : [a,b] \to \mathbf{R}$ be bounded and let P be a partition of $[a,b]$. Prove that $L(f;P) \leq U(f;P)$.

1.2. (*Proves Theorem 1.9*) Let $f : [a,b] \to \mathbf{R}$ be bounded, let $\mathcal{P}$ be the set of all partitions of $[a,b]$, let $\mathcal{L} = \{L(f;P) \mid P \in \mathcal{P}\}$, and let $\mathcal{U} = \{U(f;P) \mid P \in \mathcal{P}\}$.

(a) For $P, Q \in \mathcal{P}$, prove that $L(f;P) \leq U(f;Q)$. (Suggestion: Use common refinements.)

(b) Prove that

$$L(f;P) \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq U(f;P). \tag{1.17}$$

for any $P \in \mathcal{P}$. (Suggestion: The middle inequality is the interesting part; by part (a), every upper sum is an upper bound for $\mathcal{L}$ and every lower sum is a lower bound for $\mathcal{U}$.)

1.3. (*Proves Lemma 1.11*) For $c \in \mathbf{R}$, prove that the constant function $f(x) = c$ is integrable, and prove that $\displaystyle\int_a^b c\,dx = c(b-a)$.

1.4. Define $f : [0,1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational}, \\ 0 & \text{if } x \text{ is irrational}. \end{cases} \tag{1.18}$$

Prove that f is not integrable. (Suggestion: Prove that all upper Riemann sums $U(f;P)$ are equal and that all lower Riemann sums $L(f;P)$ are equal.)

2 The Riemann integral: Properties

In this section, we prove the "ordinary" properties of the integral, that is, the ones that are not directly related to the Fundamental Theorem of Calculus. We begin with some useful estimates.

Lemma 2.1. *Let $f, g : [a, b] \to \mathbf{R}$ be bounded, let P be a partition of $[a, b]$, and $c > 0$. Then, in the notation of Definition 1.4, we have:*

$$\begin{aligned} m(f; P, i) + m(g; P, i) &\leq m(f + g; P, i) \\ &\leq M(f + g; P, i) \leq M(f; P, i) + M(g; P, i), \end{aligned} \tag{2.1}$$

$$cm(f; P, i) = m(cf; P, i) \leq M(cf; P, i) = cM(f; P, i), \tag{2.2}$$

$$m(-f; P, i) = -M(f; P, i) \leq -m(f; P, i) = M(-f; P, i). \tag{2.3}$$

Furthermore, if $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$M(f; P, i) \leq M(g; P, i). \tag{2.4}$$

Proof. We prove only (2.1), and leave the other estimates as Problem 2.1.

Let $[x_{i-1}, x_i]$ be the ith subinterval of P. By the definitions of $M(f; P, i)$ and $M(g; P, i)$, for any $x \in [x_{i-1}, x_i]$, we have $f(x) + g(x) \leq M(f; P, i) + M(g; P, i)$. In other words, $M(f; P, i) + M(g; P, i)$ is an upper bound for $S = \{f(x) + g(x) \mid x \in [x_{i-1}, x_i]\}$. Therefore, since $M(f + g, P, i) = \sup S$ is the *least* upper bound of S, we have that $M(f + g; P, i) \leq M(f; P, i) + M(g; P, i)$. The last inequality follows, and the first inequality follows by an analogous argument. □

Theorem 2.2. *Let $f, g : [a, b] \to \mathbf{R}$ be bounded and integrable on $[a, b]$ and $c > 0$. Then $f + g$, cf, and $-f$ are integrable on $[a, b]$, and*

$$\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx, \tag{2.5}$$

$$\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx, \tag{2.6}$$

$$\int_a^b (-f(x))\, dx = -\int_a^b f(x)\, dx. \tag{2.7}$$

Furthermore, if $f(x) \leq g(x)$ for all $x \in [a, b]$,

$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx. \tag{2.8}$$

The basic idea for each property is to sum Lemma 2.1 over all of the subintervals of a partition to obtain the theorem for Riemann sums, and then take a limit of a well-chosen sequence of Riemann sums.

Proof. By Lemma 1.10, there exist sequences Q_n, Q'_n of partitions of $[a,b]$ such that

$$\lim_{n\to\infty}(U(f;Q_n)-L(f;Q_n)) = \lim_{n\to\infty}(U(g;Q'_n)-L(g;Q'_n)) = 0. \tag{2.9}$$

In fact, by taking the common refinement $P_n = Q_n \cup Q'_n$, we may replace both Q_n and Q'_n with P_n, since refinements only make these sequences converge faster (Lemma 1.8).

Turning first to (2.5), summing (2.1) over all subintervals of P_n, we see that

$$L(f;P_n)+L(g;P_n) \le L(f+g;P_n) \le U(f+g;P_n) \le U(f;P_n)+U(g;P_n). \tag{2.10}$$

It follows that

$$0 \le U(f+g;P_n)-L(f+g;P_n) \le (U(f;P_n)-L(f;P_n)) + (U(g;P_n)-L(g;P_n)), \tag{2.11}$$

and since the right-hand side converges to 0, the Squeeze Lemma implies that

$$\lim_{n\to\infty}(U(f+g;P_n)-L(f+g;P_n)) = 0. \tag{2.12}$$

Therefore, by Lemma 1.10, $f+g$ is integrable. Furthermore, since $U(f+g;P_n) \le U(f;P_n) + U(g;P_n)$ for all n, taking the $\lim\limits_{n\to\infty}$ of both sides, we see that $\displaystyle\int_a^b (f(x)+g(x))\,dx \le \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$; and since $L(f;P_n)+L(g;P_n) \le L(f+g;P_n)$ for all n, $\displaystyle\int_a^b f(x)\,dx + \int_a^b g(x)\,dx \le \int_a^b (f(x)+g(x))\,dx$. Equation (2.5) follows.

Similar arguments (Problem 2.2) give the other integrability results and formulas. ☐

Now, up to this point, we have not really used the full flexibility of allowing arbitrary (uneven) partitions. It is precisely in the proof of the next result where we see that come to fruition.

Theorem 2.3. *For $a<b<c$, let $f:[a,c]\to\mathbf{R}$ be integrable on $[a,b]$ and $[b,c]$. Then f is integrable on $[a,c]$ and*

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx. \tag{2.13}$$

Proof. See Problem 2.3. ☐

Next, uniform continuity yields the following result.

Theorem 2.4. *Let $f:[a,b]\to\mathbf{R}$ be continuous. Then f is integrable on $[a,b]$.*

Proof. See Problem 2.4. ☐

Theorem 2.5. *If $f,g:[a,b]\to\mathbf{R}$ are integrable, then the functions $|f(x)|$, $\min(f(x),g(x))$, and $\max(f(x),g(x))$ are also integrable on $[a,b]$. Furthermore,*

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx. \tag{2.14}$$

It may be helpful to think of (2.14) as an integral analogue of the triangle inequality, in that the triangle inequality can be used to give an upper bound to the absolute value of a sum, whereas Theorem 2.5 is used to give an upper bound to the absolute value of an integral.

Proof. See Problems 2.5 and 2.6. □

Problems

2.1. (*Proves Lemma 2.1*) Let $f, g : [a, b] \to \mathbf{R}$ be bounded, let P be a partition of $[a, b]$, and let $c \in \mathbf{R}$ be positive.

(a) Prove that $cm(f; P, i) = m(cf; P, i) \leq M(cf; P, i) = cM(f; P, i)$.

(b) Prove that $m(-f; P, i) = -M(f; P, i) \leq -m(f; P, i) = M(-f; P, i)$.

(c) Now assume that $f(x) \leq g(x)$ for all $x \in [a, b]$. Prove that $M(f; P, i) \leq M(g; P, i)$.

2.2. (*Proves Theorem 2.2*) Let $f, g : [a, b] \to \mathbf{R}$ be bounded and integrable and let $c \in \mathbf{R}$ be positive.

(a) Prove that cv is integrable and $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

(b) Prove that $-v$ is integrable and $\displaystyle\int_a^b (-f(x))\,dx = -\int_a^b f(x)\,dx$.

(c) Now assume that $f(x) \leq g(x)$ for all $x \in [a, b]$. Prove that $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

2.3. (*Proves Theorem 2.3*) (Suggestion: Use Lemma 1.10 and the fact that if P is a partition of $[a, b]$ and Q is a partition of $[b, c]$, then $P \cup Q$ is a partition of $[a, c]$.)For $a < b < c$ in $\mathbf{R}$, let $f : [a, c] \to \mathbf{R}$ be integrable on $[a, b]$ and $[b, c]$. Prove that f is integrable on $[a, c]$ and

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx. \tag{2.15}$$

2.4. (*Proves Theorem 2.4*) Let $f : [a, b] \to \mathbf{R}$ be continuous.

(a) Suppose f is continuous on $[x_0, x_1]$ and satisfies the condition that for $x, y \in [x_0, x_1]$, we have that $|f(x) - f(y)| < \epsilon_0$. Prove that if

$$M = \sup \{f(x) \mid x \in [x_0, x_1]\},$$
$$m = \inf \{f(x) \mid x \in [x_0, x_1]\},$$

then $|M - m| < \epsilon_0$. (Suggestion: Extreme Value Theorem.)

(b) Use the uniform continuity of f to show that given $\epsilon > 0$, for sufficiently large n, if P_n is the nth standard partition of $[a, b]$ (Example 1.2), then for x and y contained in the same subinterval of P_n, we have $|f(x) - f(y)| < \dfrac{\epsilon}{(b-a)}$.

(c) Prove that f is integrable on $[a, b]$. (Suggestion: Use Lemma 1.10.)

2.5. (*Proves Theorem 2.5*) Let $f : [a, b] \to \mathbf{R}$ be integrable.

(a) Prove that for $c, d \in \mathbf{R}$, $||c| - |d|| \leq |c - d|$.

(b) Let P be a partition of $[a, b]$. Prove that

$$M(|f|\,; P, i) - m(|f|\,; P, i) \leq M(f; P, i) - m(f; P, i). \tag{2.16}$$

(Suggestion: Arbitrarily Close Criterion.)

(c) Prove that $|f|$ is integrable. (Suggestion: Sequential Criterion for Integrability.)

2.6. (*Proves Theorem 2.5*) Let $f, g : [a, b] \to \mathbf{R}$ be integrable.

(a) Prove that for $c, d \in \mathbf{R}$, $\max(c, d) = \frac{1}{2}(c + d + |c - d|)$, and find and prove a similar formula for $\min(c, d)$.

(b) Prove that $\min(f(x), g(x))$ and $\max(f(x), g(x))$ are integrable on $[a, b]$.

(c) Prove that

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\ dx. \tag{2.17}$$

(Suggestion: Consider $f_+(x) = \max(f(x), 0)$ and $f_-(x) = \max(-f(x), 0)$, and express both sides of (2.17) in terms of f_+ and f_-.)