Partitions

Definition (Partition)

A **partition** P of $[a, b]$ is a finite subset $\{x_0, \ldots, x_n\} \subset [a, b]$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.

Terminology and abbreviations:

- $[x_{i-1}, x_i]$ is i**th subinterval** of P
- $(\Delta x)_i = x_i - x_{i-1}$
- R is a **refinement** of P means $P \subseteq R$
- $P \cup Q$ is the **common refinement** of P and Q

Upper and lower Riemann sums

Assume $f : [a, b] \to \mathbf{R}$ bounded.

Definition (Upper and lower Riemann sums)

$P = \{x_0, \ldots, x_n\}$ a partition of $[a, b]$. Define

$$M(f; P, i) = \sup \{f(x) \mid x \in [x_{i-1}, x_i]\},$$
$$m(f; P, i) = \inf \{f(x) \mid x \in [x_{i-1}, x_i]\}.$$

Upper Riemann sum $U(f; P)$ is

$$U(f; P) = \sum_{i=1}^{n} M(f; P, i)(\Delta x)_i.$$

Lower Riemann sum $L(f; P)$ is

$$L(f; P) = \sum_{i=1}^{n} m(f; P, i)(\Delta x)_i.$$

Upper and lower integrals

$\mathcal{P}$: set of all partitions of $[a, b]$.

Definition (Upper and lower Riemann integrals)

Upper Riemann integral and **lower Riemann integral** of f on $[a, b]$ are

$$\overline{\int_a^b} f(x)\,dx = \inf\{U(f;P) \mid P \in \mathcal{P}\},$$
$$\underline{\int_a^b} f(x)\,dx = \sup\{L(f;P) \mid P \in \mathcal{P}\},$$

respectively.

Upper Riemann integral is “best (lowest) possible overestimate” of area under curve, Lower Riemann integral is “best (highest) possible underestimate.”

The Riemann integral

Definition (Riemann integral)

To say f is **integrable** on $[a, b]$ means that f is bounded on $[a, b]$ and upper and lower integrals of f on $[a, b]$ are equal.
If f is integrable, **Riemann integral of f on** $[a, b]$ is

$$\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$$

Sequential Criteria for Integrability

For bounded $f : [a, b] \to \mathbf{R}$, the following are equivalent:

1. f is integrable on $[a, b]$.
2. There exists a sequence of partitions P_n such that $\lim_{n\to\infty}(U(f; P_n) - L(f; P_n)) = 0$.
3. For any $\epsilon > 0$, there exists a partition P such that $U(f; P) - L(f; P) < \epsilon$.

Furthermore, if condition (2) holds, then

$$\lim_{n\to\infty} L(f; P_n) = \int_a^b f(x)\, dx = \lim_{n\to\infty} U(f; P_n).$$