Math 131A
Fractionology and asymptotics

There are two technical skills used in computing and proving limits (and later, sums of series) that you will need repeatedly this semester: fractionology and asymptotics.

Fractionology. Suppose that $f(n)$ and $g(n)$ are functions of n, and you want to prove that $\frac{f(n)}{g(n)} < \epsilon$ for large enough n (perhaps as part of proving that $\lim \frac{f(n)}{g(n)} = 0$). Often this can be achieved by finding some $f_1(n)$, $g_1(n)$ with two properties:

- $\frac{f(n)}{g(n)} \leq \frac{f_1(n)}{g_1(n)}$; and
- It is algebraically (relatively) easy to prove that $\frac{f_1(n)}{g_1(n)} < \epsilon$.

For in that case, we will be able to show that

$$\frac{f(n)}{g(n)} \leq \frac{f_1(n)}{g_1(n)} < \epsilon.$$

Again, the point is that we want the new fraction $\frac{f_1(n)}{g_1(n)}$ to be bigger, but simpler; in other words, we are looking for some simpler $\frac{f_1(n)}{g_1(n)}$ that will lie between $\frac{f(n)}{g(n)}$ and ϵ.

The key points to remember are:

- A larger numerator makes a larger fraction, and a smaller numerator makes a smaller fraction.
- A larger denominator makes a *smaller* fraction, and a smaller denominator makes a *larger* fraction.

For example,

$$\begin{aligned} \frac{n^2+1}{2n^4+5} &\leq \frac{n^2+1}{2n^4} && \text{(smaller denominator)} \\ &\leq \frac{n^2+n^2}{2n^4} && \text{(larger numerator)} \\ &= \frac{2n^2}{2n^4} = \frac{1}{n^2}. \end{aligned}$$

On the other hand, for $n \geq 2$, we have that $3 < n^3$, and $n^3 < 2n^3 - 3$. Therefore, for $n \geq 2$,

$$\begin{aligned} \frac{n^2-7}{2n^3-3} &\leq \frac{n^2}{2n^3-3} && \text{(larger numerator)} \\ &\leq \frac{n^2}{n^3} && \text{(smaller denominator if } n \geq 2) \\ &= \frac{1}{n}. \end{aligned}$$

Asymptotics. It will also be useful to be able to compare the rates of growth of various sequences as $n \to \infty$ (The Asymptotic Theorem, below). To state the main result, we use the following idea.

Definition. Suppose a_n and b_n are positive-valued sequences. To say that $a_n << b_n$ means that

$$\lim \frac{a_n}{b_n} = 0.$$

Exercise. Prove that $<<$ is transitive, i.e., prove that if $a_n << b_n$ and $b_n << c_n$, then $a_n << c_n$.

We will not be able to prove our main result (The Asymptotics Theorem) completely rigorously because we will not have proven the necssary calculus until the end of the course. However, the theorem will be useful in considering examples, so for now, we list some properties of exponential and log functions on which we will rely for now, and prove later.

- We assume that the usual algebraic properties of a^b and $\ln x$ hold, including the fact that $\ln(e^x) = x$ for $x \in \mathbb{R}$.
- We assume that if $0 < a < b$, then $\ln a < \ln b$ (i.e., $\ln x$ is increasing.)
- We assume that if (x_n) is a convergent sequence such that $x_n > 0$ and $\lim x_n > 0$, then $\lim(\ln(x_n)) = \ln(\lim x_n)$. (This condition may look strange, but as we shall soon see, this says precisely that $\ln x$ is continuous for $x > 0$.)

Theorem (The Asymptotics Theorem)**.** *For fixed $p > 0$ and $a > 1$, we have that*

$$1 << \ln n << n^p << a^n << n!.$$

Proof. Exercise 9.14 implies that $n^p << a^n$, and Exercise 9.15 implies that $a^n << n!$, so it remains to show that $1 << \ln n$ and $\ln n << n^p$.

To see that $1 << \ln n$, it suffices to show that $\lim(\ln n) = +\infty$, for then Theorem 9.10 implies that $\lim \left(\frac{1}{\ln n} \right) = 0$. For $M \in \mathbb{R}$, let $N(M) = e^M$. If $n \in \mathbb{Z}$, $n > N(M)$, then

$$\ln n > \ln e^M = M.$$

To see that $\ln n << n^p$ for $p = 1$, we begin with the fact that $\lim n^{1/n} = 1$ (Theorem 9.7(c)). Then

$$\lim \left(\frac{\ln n}{n} \right) = \lim \left(\ln(n^{1/n}) \right) = \ln \left(\lim(n^{1/n}) \right) = \ln 1 = 0.$$

A related, but more involved, argument actually works in general. However, that argument requires much more calculus, so we omit it. □

For example, since $1 << n^{1/2} = \sqrt{n}$, we have that

$$\lim \frac{1}{\sqrt{n}} = 0;$$

since $\ln n << n^{1/3}$, we have that

$$\lim \frac{\ln n}{\sqrt[3]{n}} = 0;$$

and since $10^n << n!$, we have that

$$\lim \frac{10^n}{n!} = 0.$$