Real Analysis I, summarized in 16 pages

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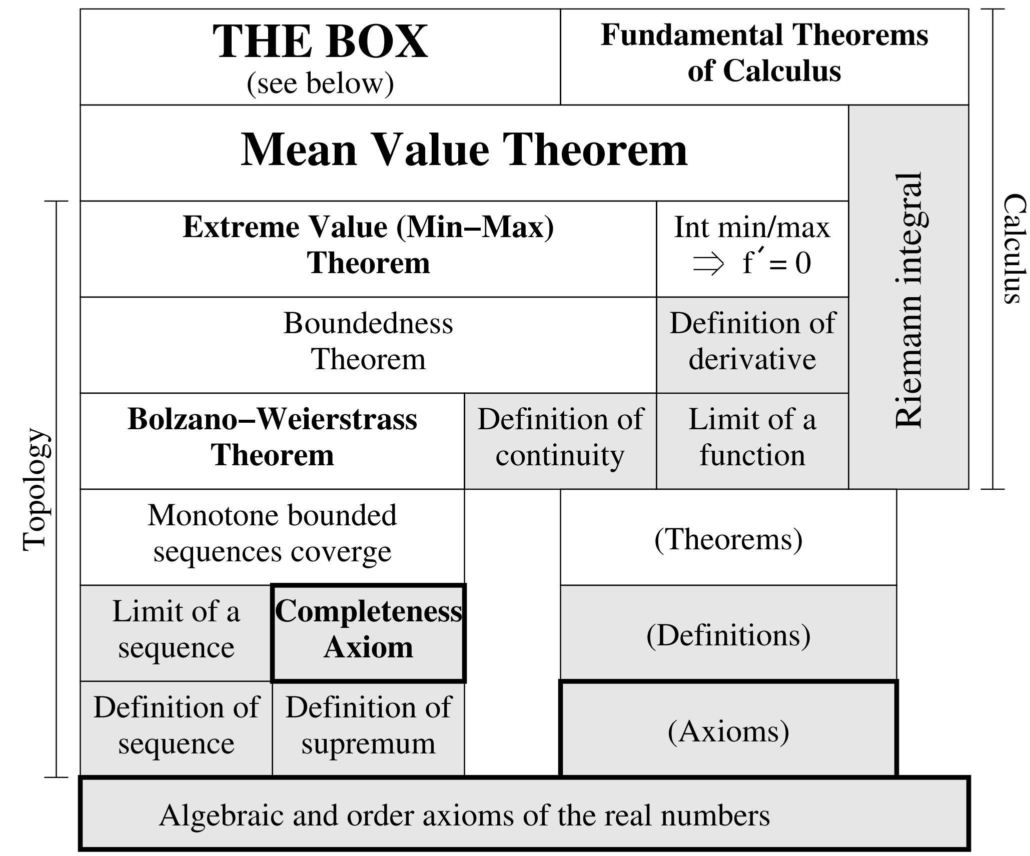
The following picture shows the logical structure of the main axioms, definitions, and theorems in Analysis I. Theorems are in unshaded boxes, definitions are in shaded lighter boxes, and axioms are in shaded heavier boxes. More important results are in boldface or larger type.



The summary

We assume that the real numbers $\mathbf{R}$ satisfy the axioms of an ordered field, i.e., grade school arithmetic and $\leq$ work the way you think they do. To describe the one other key axiom we assume for $\mathbf{R}$, we need the following definitions.

Definition (Upper bound)**.** Let S be a nonempty subset of $\mathbf{R}$. To say that $u \in \mathbf{R}$ is an **upper bound** for S means that for every $x \in S$, we have $x \leq u$.

Definition (Supremum)**.** Let S be a nonempty subset of $\mathbf{R}$. To say that $u \in \mathbf{R}$ is a **supremum** for S, or $u = \sup S$, means that two things hold:

1. u is an upper bound for S.

2. u is the **least** upper bound of S.

Part 2 of the definition of supremum has two equivalent reformulations:

$2'$. If v is an upper bound for S, then $u \le v$.

$2''$. If $v < u$, then v is not an upper bound for S.

Version $2''$ means that you can use the following structure to prove that an upper bound u for S is the **least** upper bound of S:

Assume $v < u$. $\vdots$ (stuff) $\vdots$ (*) So there exists some $x \in S$ such that $v < x$.

The point being, (*) says precisely that v is not an upper bound for S.
In any case, we can now state the last axiom we assume for $\mathbf{R}$:

Axiom (Completeness/Least upper bound property)**.** Let S be a nonempty subset of $\mathbf{R}$. If S is bounded above (has an upper bound), then $\sup S$ exists.

There are analogous definitions of lower bound and $\inf S$, the greatest lower bound of S. The least upper bound property also implies an analogous greatest lower bound property: Every nonempty subset of $\mathbf{R}$ that is bounded below has an inf. We also have the following useful tool for interpreting the meaning of $\sup S$.

Theorem (Arbitrarily Close Criterion)**.** *Let S be a nonempty subset of $\mathbf{R}$, and suppose that u is an upper bound for S. Then the following are equivalent:*

1. *For every $\epsilon > 0$, there exists some $s \in S$ such that $u - s < \epsilon$ (i.e., $u - \epsilon < s \le u$).*
2. *$u = \sup S$.*

We next turn to sequences.

Definition (Sequence)**.** A sequence in a set $X \subseteq \mathbf{R}$ is a function $a : \mathbf{N} \to \mathbf{R}$. By convention, instead of $a(n)$, we write a_n.

In other words, a sequence is a list $a_1, a_2, a_3, \dots \in X$, in that particular order. Note that sometimes sequences begin $a_0, a_1, \dots$, or $a_{13}, a_{14}, \dots$, or $a_{-3}, a_{-2}, \dots$, etc.

Definition (Limit of a sequence)**.** For a sequence a_n in $\mathbf{R}$ and $L \in \mathbf{R}$, to say that $\lim_{n\to\infty} a_n = L$ (or a_n **converges** to L) means that for every $\epsilon > 0$, there exists some $N(\epsilon)$ such that if $n \in \mathbf{N}$ and $n > N(\epsilon)$, then $|a_n - L| < \epsilon$.

The definition of $\lim_{n\to\infty} a_n = L$ is complicated, with multiple nested layers. However, it has the virtue that every proof that $\lim_{n\to\infty} a_n = L$ has roughly the following structure:

Assume $\epsilon > 0$.
We choose $N(\epsilon) = ??$.
Assume $n \in \mathbf{N}$, $n > N(\epsilon)$.
$\vdots$
(stuff)
$\vdots$
We conclude that $|a_n - L| < \epsilon$.
So if $n \in \mathbf{N}$ and $n > N(\epsilon)$, then $|a_n - L| < \epsilon$.
So there exists some $N(\epsilon)$ such that if $n \in \mathbf{N}$ and $n > N(\epsilon)$, then $|a_n - L| < \epsilon$.
Therefore, for every $\epsilon > 0$, there exists some $N(\epsilon)$ such that if $n \in \mathbf{N}$ and $n > N(\epsilon)$, then $|a_n - L| < \epsilon$.

Again: This structure is complicated, but it's all there in the definition of limit.

We come to two theorems that form the heart of the really deep part of Analysis I. To state these, we need some definitions.

Definition (Monotone sequences)**.** Let a_n be a sequence in $\mathbf{R}$. To say that a_n is **increasing** means that for all n, $a_n \leq a_{n+1}$, and to say that a_n is **decreasing** means that for all n, $a_n \geq a_{n+1}$. Increasing and decreasing sequences are together called **monotone** sequences.

Definition. Let a_n be a sequence in $\mathbf{R}$. To say that a_n is **bounded above** means that there exists some M such that $a_n \leq M$ for all $n \in \mathbf{N}$; and to say that a_n is **bounded below** means that there exists some L such that $L \leq a_n$ for all $n \in \mathbf{N}$. To say that a_n is **bounded** means that a_n is both bounded above and bounded below; equivalently, a_n bounded exactly when there exists some M such that $|a_n| \leq M$ (i.e., $-M \leq a_n \leq M$) for all $n \in \mathbf{N}$.

Definition. Let a_n be a sequence in $\mathbf{R}$. A **subsequence** of a_n is a sequence a_{n_k} in the variable k, where $n_1 < n_2 < n_3 < \dots$ is a strictly increasing sequence in $\mathbf{N}$ (i.e., a stricly increasing sequence of indices).

Theorem (Monotone sequences converge)**.** *Let a_n be an increasing sequence in $\mathbf{R}$ that is bounded above, and let $S = \{a_n \mid n \in \mathbf{N}\}$ (i.e., S is the unordered set of values that occur in the sequence a_n). Then a_n converges to $\sup S$.*

Similarly, a decreasing sequence that is bounded below converges to the inf of the set of its values.

The monotone sequence theorem, combined with the fact that every sequence has a monotone subsequence, leads to the following result, which is the magical engine that makes the rest of Analysis I go.

Theorem (Bolzano-Weierstrass)**.** *Every bounded sequence in $\mathbf{R}$ has a convergent subsequence.*

Note: Some books/courses may express the above material in terms of the notion of *compactness*. For $E \subseteq \mathbf{R}$, to say that E is *open* means that E is a union (possibly horribly infinite) of open intervals (a, b); and to say that E is *closed* means that if a_n is a convergent sequence in E, then $\lim_{n\to\infty} a_n \in E$. To say that $E \subseteq \mathbf{R}$ is *compact* means that every (possibly horribly infinite) open cover of E (collection of open sets whose union contains E) has a finite subcover. The *Heine-Borel Theorem* then says that the $E \subseteq \mathbf{R}$ is compact if and only if E is closed and bounded. Fancy! But if you look at the details, you'll see that the proof of Heine-Borel is driven by Bolzano-Weierstrass, because that's the natural user interface for the least upper bound property.

While the following definition is not strictly necessary for establishing the foundations of analysis, it is both useful and important in the further study of analysis.

Definition (Cauchy sequence)**.** For a sequence a_n in $\mathbf{R}$ and $L \in \mathbf{R}$, to say that a_n is **Cauchy** means that for every $\epsilon > 0$, there exists some $N(\epsilon)$ such that if $n, k \in \mathbf{N}$ and $n, k > N(\epsilon)$, then $|a_n - a_k| < \epsilon$.

In other words, the terms of a Cauchy sequence get closer to each other, but not necessary to a fixed limit L. However, it follows from (for example) Bolazno-Weierstrass that:

Theorem (Cauchy completeness of $\mathbf{R}$)**.** *Every Cauchy sequence in* $\mathbf{R}$ *converges to some limit in* $\mathbf{R}$.

Note: The property of "every Cauchy sequence in X has a limit in X", stated above for $X = \mathbf{R}$, is a way to generalize the completeness or $\mathbf{R}$ to other settings.

We now turn to real-valued functions defined on intervals in $\mathbf{R}$.

Definition (Continuity)**.** Let X be a subset of $\mathbf{R}$ and let $f : X \to \mathbf{R}$ be a real-valued function on X. To say that f is **continuous** at $a \in X$ means that one of the following conditions holds:

1. **(Epsilon-delta continuity)** For every $\epsilon > 0$, there exists some $\delta(\epsilon) > 0$ such that if $x \in X$ and $|x - a| < \delta(\epsilon)$, then $|f(x) - f(a)| < \epsilon$.

2. **(Sequential continuity)** For every sequence x_n in X with $\lim_{n\to\infty} x_n = a$, we have that $\lim_{n\to\infty} f(x_n) = f(a)$.

To say that f is **continuous on** X means that for every $a \in X$, f is continuous at a.

It can be shown that the two versions of the definition of continuity are equivalent, and each of them has its uses and its own associated proof structure. For example, to prove that $f : X \to \mathbf{R}$ is continuous at a using the epsilon-delta definition:

Assume $\epsilon > 0$.

  We choose $\delta(\epsilon) = ??$.

    Assume $x \in X$, $|x - a| < \delta(\epsilon)$.

    $\vdots$

    (stuff)

    $\vdots$

    We conclude that $|f(x) - f(a)| < \epsilon$.

  So if $x \in X$, $|x - a| < \delta(\epsilon)$, then $|f(x) - f(a)| < \epsilon$.

So there exists some $N(\epsilon)$ such that if $x \in X$ and $|x - a| < \delta(\epsilon)$, then $|f(x) - f(a)| < \epsilon$.

Therefore, for every $\epsilon > 0$, there exists some $N(\epsilon)$ such that if $x \in X$ and $|x - a| < \delta(\epsilon)$, then $|f(x) - f(a)| < \epsilon$.

To prove that $f : X \to \mathbf{R}$ is continuous at a using the sequential definition:

Assume x_n is a sequence in X and $\lim_{n\to\infty} x_n = a$.

    $\vdots$

    (stuff)

    $\vdots$

    We conclude that $\lim_{n\to\infty} f(x_n) = f(a)$.

And yes, at first, the sequential structure looks much simpler! However, to prove that $\lim_{n\to\infty} f(x_n) = f(a)$, you may have to use the ϵ-$N(\epsilon)$ definition of the limit and set up a nested structure as before. Furthermore, the assumption that $\lim_{n\to\infty} x_n = a$ itself is an ϵ-$N(\epsilon)$ statement like "For every $\epsilon_1 > 0$, there exists…" that must be unlocked by choosing a suitable ϵ_1. In other words, the sequential definition of the limit doesn't really eliminate the work in proofs about continuity; it just hides that work under a layer of abstraction.

Sequential continuity does have some notable advantages, though. For example, limit laws (e.g., the sum law for limits) directly imply laws of continuity (e.g., the sum of continuous functions is continuous). To give another example, the fact that the composition of continuous functions is continuous follows quite directly and naturally from the sequential definition. Finally, because the negation of the epsilon-delta is quite awkward to work with, often the best way to prove that a function is not continuous is to use the **bad sequence** method:

To prove that $f : X \to \mathbf{R}$ is *not* continuous at $x = c$, find a sequence x_n in X such that:

- $\lim_{n\to\infty} x_n = c$, but
- $\lim_{n\to\infty} f(x_n) \neq f(c)$.

Note that the second condition on a bad sequence can be proven either by proving that $\lim_{n\to\infty} f(x_n)$ does not exist or by proving that $\lim_{n\to\infty} f(x_n)$ exists but is not equal to $f(c)$.

While continuous functions have numerous interesting properties, the following is the most important one for calculus. Call a real-valued function f **bounded** on X if there exists some $M > 0$ such that $|f(x)| \leq M$ for all $x \in X$.

Theorem (Extreme Value Theorem)**.** *Let f be a real-valued function that is continuous on a closed and bounded interval $[a, b]$. Then f is bounded and f attains both minimum and maximum values on $[a, b]$. In other words, there exist $c, d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.*

We also have the following theorem, which is not as pivotal as the Extreme Value Theorem, but is still important.

Theorem (Intermediate Value Theorem)**.** *Let f be a real-valued function that is continuous on an interval I. Suppose $a < b$ in I. If $f(a) < f(b)$, then for any $y \in \mathbf{R}$ such that $f(a) < y < f(b)$, there exists $c \in I$ such that $a < c < b$ and $f(c) = y$; and if $f(a) > f(b)$, then for any $y \in \mathbf{R}$ such that $f(a) > y > f(b)$, there exists $c \in I$ such that $a < c < b$ and $f(c) = y$.*

There are several common approaches to proving the Extreme Value Theorem and Intermediate Value Theorem. If we choose to take a relatively elementary and direct approach, we can use the sequential definition of limit and various consequences of completness (sups and Bolzano-Weierstrass) to prove those theorems. More abstractly, we can use the ideas of compactness and continuity to show that if f is continuous, then the image of a compact set under f is compact, and that the image of an interval under f is an interval. The Extreme Value Theorem then follows from the Heine-Borel Theorem (see above) and the Intermediate Value Theorem follows from the "between-ness" property of intervals.

We next have the idea of the limit of a function at a point.

Definition (Limit)**.** Let X be a subset of $\mathbf{R}$ and let $f : X \to \mathbf{R}$ be a real-valued function on X. $\boxed{\text{Let } a \text{ be a limit point of } X}$, i.e., suppose a is the limit of some sequence in X but not necessarily itself in X. To say that $\lim\limits_{x \to a} f(x) = L$ means that one of the following conditions holds:

1. **(Epsilon-delta limit)** For every $\epsilon > 0$, there exists some $\delta(\epsilon) > 0$ such that if $x \in X$, $\boxed{x \neq a}$, and $|x - a| < \delta(\epsilon)$, then $|f(x) - L| < \epsilon$.

2. **(Sequential limit)** For every sequence x_n in X with $\lim\limits_{n \to \infty} x_n = a$ $\boxed{\text{and } x_n \neq a}$, we have that $\lim\limits_{n \to \infty} f(x_n) = L$.

The definition of limit is so similar to the definition of continuity that the main differences are really only those boxed above, plus the fact that we replace $f(a)$ with the limit value L. (I.e., with limits, we ignore what happens exactly *at* a and only worry about what happens *near* a.) Consenquently:

- As with continuity, the epsilon-delta and sequential definitions can be shown to be logically equivalent.
- From the sequential definition, we get constant multiple, sum, product, and quotient laws for limits, in the same way we get laws of continuity.
- A function f is continuous at $x = a$ exactly when $\lim_{x \to a} f(x) = f(a)$.
- The epsilon-delta and sequential proof strategies for limits are the same as their analogues for continuity, except that we assume $x \neq a$ or $x_n \neq a$ at the appropriate junctures.

As with continuity, sequential methods are also often useful to prove that a limit does not exist, though they are more complicated than in the case of continuity, as we need to rule out the possibility of *any* limit L existing, and not just one.

For $f : X \to \mathbf{R}$, to prove that $\lim_{x \to c} f(x)$ does not exist, either:

- (**Really bad sequence:**) Find a sequence x_n in X such that:
  - $\lim_{n \to \infty} x_n = c$ and $x_n \neq c$, but
  - $\lim_{n \to \infty} f(x_n)$ does not exist.
- (**Conflicting sequences:**) Find sequences x_n, x'_n in X such that:
  - $\lim_{n \to \infty} x_n = c$ and $x_n \neq c$,
  - $\lim_{n \to \infty} x'_n = c$ and $x'_n \neq c$, but
  - $\lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(x'_n)$.

The main reason we need function limits is the following definition.

Definition (Derivative)**.** Let I be an interval, $f : I \to \mathbf{R}$ a function, and $a \in I$. To say that f is **differentiable** at a means that the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{1}$$

exists. If $f'(a)$ exists, we call it the **derivative of** f **at** a, and if we let the point of differentiability vary, we get a function $f'(x)$, called the **derivative of** f.

We can use our previous results about limits of functions to recover the usual algebraic and composition properties of derivatives: the constant multiple, sum, product, quotient, and chain rules, with the chain rule requiring more thought than the others. Also, differentiable functions are "nicer" than continuous functions:

Theorem (Differentiability implies continuity)**.** *Let I be an interval, $f : I \to \mathbf{R}$ a function, and $a \in I$. If f is differentiable at a, then f is continuous at a.*

In terms of subequent theory, however, the key result is the Mean Value Theorem (MVT). To reach the MVT, we start by analyzing local minima/maxima and derivatives.

Definition (Local minima and maxima)**.** Let I be an interval in $\mathbf{R}$, and $f : I \to \mathbf{R}$ a function. To say that f has a **local maximum** at $c \in I$ means that there exists some $\delta > 0$ such that if $x \in I$ and $|x - c| < \delta$, then $f(x) \leq f(c)$. Similarly, to say that f has a **local minimum** at $c \in I$ means that there exists some $\delta > 0$ such that if $x \in I$ and $|x - c| < \delta$, then $f(c) \leq f(x)$.

Note that the above definition of local minimum/maximum differs from the definition in many calculus textbooks, because our defintion allows f to have a local min or max at an endpoint of its domain, whereas many calculus textbooks specifically rule out endpoints as local maxima or minima.

Theorem (Fermat's Theorem)**.** *If $f : [a, b] \to \mathbf{R}$ has a local minimum or maximum at $c \in (a, b)$ (i.e., c is not an endpoint) and f is differentiable at c, then $f'(c) = 0$.*

Fermat's Theorem plus the Extreme Value Theorem then yield the following theorem, which in many ways is the summit of differential calculus.

Theorem (Mean Value Theorem)**.** *Let $f : [a, b] \to \mathbf{R}$ be differentiable on (a, b) (i.e., except possibly at endpoints) and continuous on $[a, b]$. Then there exists some $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{2}$$

In other words, under the hypotheses of the MVT, there exists some point in the interior of the domain of f where the instantaneous rate of change is equal to the average rate of change over the whole interval $[a, b]$. Note that MVT follows from the special case where $f(a) = f(b)$, in which case the MVT says that there exists some $c \in (a, b)$ such that $f'(c) = 0$ (a result also known as *Rolle's Theorem*).

There are many consequences of the MVT, but perhaps the most notable one is **THE BOX**, which summarizes Calculus I in a box:

f	$\nearrow$	$\searrow$	$\smile$	$\frown$
f'	$+$	$-$	$\nearrow$	$\searrow$
		f''	$+$	$-$

In other words, if $f'(x) > 0$ for all x in some open interval, then f is strictly increasing on that interval; and so on.

We come to the Riemann integral, whose definition in analysis differs from the one typically seen in calculus in one key aspect: Instead of only considering Riemann sums based on uniform partitions with $\Delta x = (b - a)/n$ (Figure 1), we also consider Riemann sums based on partitions with variable Δx_i (Figure 2). This greater flexibility is ultimately helpful in proving the properties of the definite integral, but it does require more complicated definitions, as follows.

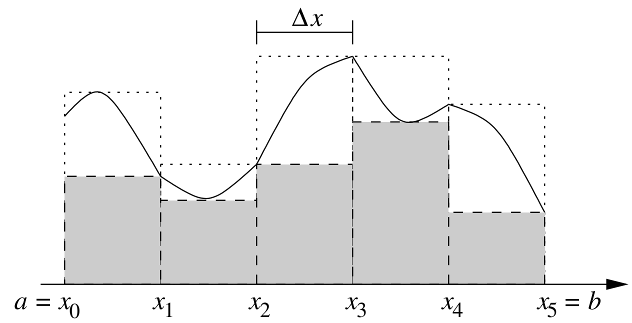


Figure 1: Riemann sums with a uniform partition

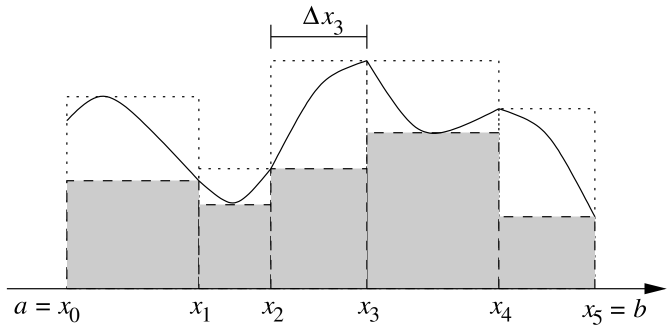


Figure 2: Riemann sums with a nonuniform partition

Definition (Partition)**.** A **partition** P of $[a, b]$ is a finite subset $\{x_0, \ldots, x_n\} \subset [a, b]$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. We call $[x_{i-1}, x_i]$ the i**th subinterval** of P, and when P is understood, we use the abbreviation $(\Delta x)_i = x_i - x_{i-1}$. See Figure 2.

Suppose P and Q are partitions of $[a, b]$. If $P \subseteq Q$, we say that Q is a **refinement** of P, as each subinterval of P is the union of of one or more subintervals of Q. Similarly, we use $P \cup Q$ to denote the partition of $[a, b]$ obtained from the points of $P \cup Q$, written in ascending order, and we call $P \cup Q$ the **common refinement** of P and Q.

Definition (Upper and lower Riemann sums)**.** Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. Since we continue to assume that $f(x)$ is bounded, we can define

$$\begin{aligned} M(f; P, i) &= \sup \{f(x) \mid x \in [x_{i-1}, x_i]\}, \\ m(f; P, i) &= \inf \{f(x) \mid x \in [x_{i-1}, x_i]\}. \end{aligned} \tag{3}$$

We define the **upper Riemann sum** $U(f; P)$ to be

$$U(f; P) = \sum_{i=1}^{n} M(f; P, i)(\Delta x)_i, \tag{4}$$

and we define the **lower Riemann sum** $L(f; P)$ to be

$$L(f; P) = \sum_{i=1}^{n} m(f; P, i)(\Delta x)_i. \tag{5}$$

In other words, we define Riemann sums as in calculus, except that we use sups and infs instead of values of $f(x)$, and $(\Delta x)_i$ is variable, not constant. Again, see Figure 2.

Definition (Upper and lower Riemann integrals)**.** Let $\mathcal{P}$ be the set of all partitions of $[a, b]$. We define the **upper Riemann integral** and **lower Riemann integral** of f on $[a, b]$ to be

$$\overline{\int_a^b} f(x)\,dx = \inf\left\{U(f;P) \mid P \in \mathcal{P}\right\}, \tag{6}$$

$$\underline{\int_a^b} f(x)\,dx = \sup\left\{L(f;P) \mid P \in \mathcal{P}\right\}, \tag{7}$$

respectively. Note that (at least in pictures, see Figure 2) each $U(f;P)$ is an overestimate of the area under $y = f(x)$ and each $L(f;P)$ is an underestimate of the area under $y = f(x)$, so we may think of (6) and (7) as best possible upper and lower estimates to the area under the curve, respectively.

Definition (Riemann integral)**.** To say that f is **integrable** on $[a, b]$ means that f is bounded on $[a, b]$ and the upper and lower integrals of f on $[a, b]$ are equal. If f is integrable, we define the **Riemann integral of f on** $[a, b]$ to be

$$\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx. \tag{8}$$

To prove that any particular function or class of functions is integrable, we need some kind of technical tool like the following.

Theorem (Sequential Criteria for Integrability)**.** *Let $f : [a, b] \to \mathbf{R}$ be bounded. Then the following are equivalent.*

1. *f is integrable on $[a, b]$.*
2. *There exists a sequence of partitions P_n such that $\lim_{n\to\infty} (U(f;P_n) - L(f;P_n)) = 0$.*
3. *For any $\epsilon > 0$, there exists a partition P such that $U(f;P) - L(f;P) < \epsilon$.*

Furthermore, if condition (2) holds, then

$$\lim_{n\to\infty} L(f;P_n) = \int_a^b f(x)\,dx = \lim_{n\to\infty} U(f;P_n). \tag{9}$$

Most notably, one can use the sequential criterion to show that continuous functions are integrable. We therefore see that:

$$\text{Differentiability} \Rightarrow \text{Continuity} \Rightarrow \text{Integrability}$$

The converses of those implications do not hold, as, for example, $f(x) = |x|$ is continuous but not differentiable at 0, and nonconstant step functions are integrable but not continuous.

We can also use the sequential criterion to prove the usual properties of the definite integral from calculus, like **linearity**:

$$\int_a^b (cf(x) + dg(x))\,dx = c\int_a^b f(x)\,dx + d\int_a^b g(x)\,dx \tag{10}$$

and **additivity of domain**:

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx. \tag{11}$$

Note that the proof of (11) is the main reason to use nonuniform partitions, in that the union of a partition of $[a, b]$ and a partition of $[b, c]$ is a (nonuniform) partition of $[a, c]$, a fact that does not hold for uniform partitions.

Two other properties of the integral are not often used in calculus, but are frequently used in analysis.

- **Comparision:** If $f, g : [a, b] \to \mathbf{R}$ are integrable and $f(x) \leq g(x)$ for all $x \in [a.b]$, we have that
$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx. \tag{12}$$
- **Triangle inequality for integrals:** If $f : [a, b] \to \mathbf{R}$ is integrable, then so is $|f(x)|$, and
$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx. \tag{13}$$
Note the analogy with the triangle inequality, as written in the form
$$\left|\sum_{i=1}^n a_i\right| \leq \sum_{i=1}^n |a_i|\,. \tag{14}$$

The crowning achievement of first-semester analysis is the proof of the Fundamental Theorems of Calculus. Different texts number these theorems differently, so we use descriptive names.

Theorem (FTC: Integral of a derivative)**.** *Let $f : [a, b] \to \mathbf{R}$ be differentiable on (a, b) and continuous on $[a, b]$, and suppose that f' is integrable on $[a, b]$. Then*

$$\int_a^b f'(x)\,dx = f(b) - f(a). \tag{15}$$

Note that the hypotheses of the Mean Value Theorem reappear above, as they are precisely what is needed for the proof.

Theorem (FTC: Derivative of an integral)**.** *Let I be an interval with $a \in I$, let $f : I \to \mathbf{R}$ be integrable on any closed and bounded subinterval of I, and define*

$$F(x) = \int_a^x f(t)\,dt. \tag{16}$$

Then F is continuous on I. Furthermore, if f is continuous at $c \in I$, then F is differentiable at c, and

$$F'(c) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right)\bigg|_{x=c} = f(c). \tag{17}$$

The reader may recognize the "integral of a derivative" FTC as the main tool used in calculus to evaluate definite integrals exactly. The "derivative of an integral" FTC is also useful because it shows that, for example, every continuous function $f(x)$ has an antiderivative of the form $F(x) = \displaystyle\int_a^x f(t)\,dt$, which one can think of a proof of the existence of a solution $y = F(x)$ to the differential equation $\dfrac{dy}{dx} = f(x)$.

Infinite series

While not necessary for the development of calculus per se, **series** are often a part of a first-semester course in analysis.

Definition (Series)**.** Let a_n $(n \geq k)$ be a sequence in $\mathbf{R}$. We define the corresponding (infinite) **series** $\displaystyle\sum_{n=k}^{\infty} a_n$ as follows.

- First, we recursively define the **sequence of partial sums** s_N by setting $s_k = a_k$ and, for $N \geq k$, setting $s_{N+1} = s_N + a_{N+1}$. In other words:

$$s_N = a_k + a_{k+1} + \cdots + a_{N-1} + a_N. \tag{18}$$

- To say that $\displaystyle\sum_{n=k}^{\infty} a_n$ **converges** means that the sequence of partial sums s_N converges, and similarly for **divergence**. Furthermore, if $\displaystyle\sum_{n=k}^{\infty} a_n$ converges, we define

$$\sum_{n=k}^{\infty} a_n = \lim_{N\to\infty} s_N. \tag{19}$$

Series are just sequences of partial sums, so their convergence and divergence can be analyzed using all of the methods previously developed for sequences. However, there are several methods especially designed for series that are quite useful. For example, when a series has a less complicated formula, we can understand its convergence with the ratio and p-series tests.

Definition (Absolute convergence)**.** To say that a series $\sum a_n$ **converges absolutely** means that $\sum |a_n|$ converges. Note that it follows from the Comparison Test (below) that if $\sum |a_n|$ converges, then the original series $\sum a_n$ then also converges.

Theorem (Ratio Test)**.** *Suppose a_n is a sequence such that $a_n \neq 0$ and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$. Then:*

1. *If $r < 1$, then $\sum a_n$ converges absolutely.*
2. *If $r > 1$, then $\sum a_n$ diverges.*

Theorem (p-series)**.** *The p-**series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p \leq 1$.*

Series with more complicated formulas are often analyzed by comparing them to less complicated series whose convergence we can understand via the ratio or p-series tests. Specifically:

Corollary (Comparison Test)**.** *Let a_n and b_n be sequences, with $b_n \geq 0$.*

1. *If $\sum b_n$ converges and $|a_n| \leq b_n$ for all n (or sufficiently large n), then $\sum a_n$ converges.*
2. *If $\sum b_n$ diverges, $a_n \geq 0$, and $b_n \leq a_n$ for all n (or sufficently large n), then $\sum a_n$ diverges.*

Using the Comparison Test often involves maniuplating inequalities in a not-necessarily-intiuitive way. The following version of the Comparison Test replaces such manipulations by taking limits.

Theorem (Limit Comparison)**.** *Let $\sum a_n$ and $\sum b_n$ be series with $a_n, b_n > 0$, and suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = C$, where $0 < C < +\infty$. Then $\sum a_n$ converges if and only if $\sum b_n$ converges.*

Finally, we also have the following result, which is useful as long as the reader avoids its tempting, but false, converse.

Corollary (nth Term Test for Divergence)**.** *If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Equivalently, if $\lim_{n\to\infty} a_n$ either does not exist or has a nonzero value, then $\sum a_n$ diverges.* □

When we have a sequence of functions $f_n : X \to \mathbf{R}$, the question of whether the sequence f_n converges to some $f : X \to \mathbf{R}$, and how, can be quite subtle. First, to say that f_n converges to f **pointwise** means that for any fixed $x \in X$, we have that $\lim_{n\to\infty} f_n(x) = f(x)$. In contrast:

Definition (Uniform convergence)**.** Let $f_n : X \to \mathbf{R}$ be a sequence of functions and let $f : X \to \mathbf{R}$ be a function. To say that the sequence f_n **converges uniformly** to f on X means that for any $\epsilon > 0$, there exists some $N(\epsilon)$ *independent of* $x \in X$ such that for any $x \in X$ and $n \in \mathbf{Z}$ such that $n > N(\epsilon)$, we have $|f(x) - f_n(x)| < \epsilon$.

For comparison, to say that f_n converges pointwise to f on X means that for any $x \in X$ and any $\epsilon > 0$, there exists some $N(\epsilon, x)$ such that for any $x \in X$ and $n \in \mathbf{Z}$ such that $n > N(\epsilon, x)$, we have $|f(x) - f_n(x)| < \epsilon$. In other words, the difference between pointwise and uniform continuity is that in uniform continuity, there is some worst-case "rate of convergence" $N(\epsilon)$, independent of x, that holds for *all* $x \in X$ simultaneously.

Uniform convergence is important because the (perhaps more natural) concept of pointwise convergence does not preserve some key function properties. For example, it is possible to find a sequence of functions f_n that converges pointwise to a function f such that:

- Each f_n is continuous, but f is not continuous.
- Each f_n is differentiable, but f is not differentiable.
- Both f and each of the f_n are differentiable, but

$$\lim_{n\to\infty} f_n'(x) \neq f'(x). \tag{20}$$

- Each f_n is integrable, but f is not integrable.
- Both f and each of the f_n are integrable, but

$$\lim_{n\to\infty} \int_a^b f_n(x), dx \neq \int_a^b f(x), dx. \tag{21}$$

On the other hand, suppose f_n converges *uniformly* to f. In that case:

- If each f_n is continuous, then f is continuous.
- If each f_n is integrable, then f is integrable, and

$$\lim_{n\to\infty} \int_a^b f_n(x), dx = \int_a^b f(x), dx. \tag{22}$$

In other words, we can move the limit past the integral:

$$\lim_{n\to\infty} \int_a^b f_n(x), dx = \int_a^b \lim_{n\to\infty} f_n(x), dx. \tag{23}$$

Note that if each f_n is differentiable, even if f_n converges uniformly to f, we can still get a non-differentiable f or a situation where (20) occurs. To preserve differentiability, we need stronger conditions along the lines of "Both f_n and f_n' converge uniformly."

To determine if f_n converges uniformly:

- If $f_n(x)$ doesn't converge pointwise, it can't converge uniformly.
- If $f_n(x)$ converges pointwise to some function $f(x)$:
  - Uniform convergence preserves continuity, so if $f_n(x)$ is continuous and $f(x)$ isn't, then convergence can only be pointwise.
  - Uniform convergence preserves integrability and integrals, so if $f_n(x)$ is integrable and $f(x)$ isn't, or if (22) doesn't hold, then convergence can only be pointwise.
  - Otherwise, you may have to do things the hard way: Let

$$d_n = \sup\{|f_n(x) - f(x)| \mid x \in X\}. \tag{24}$$

    If $\lim_{n\to\infty} d_n = 0$, then convergence is uniform; otherwise, convergence can only be pointwise.

Fortunately, in the important case of a *series* of functions $\sum_{n=0}^{\infty} g_n(x)$, we have a specialized tool for proving uniform convergence.

Theorem (Weierstrass M-test)**.** *Let X be a subinterval of $\mathbf{R}$, let $g_n : X \to \mathbf{R}$ be a sequence of functions, and suppose that M_n is a sequence of nonnegative real numbers such that $\sum M_n$ converges (absolutely) and*

$$|g_n(x)| \le M_n \tag{25}$$

for all $x \in X$. Then $\sum_{n=0}^{\infty} g_n(x)$ converges absolutely and uniformly to some $f : X \to \mathbf{R}$.

Uniform convergence works particularly well in the case of a **power series** $f(x) = \sum_{n=0}^{\infty} a_n x^n$. First, the fundamental feature of power series is the **radius of convergence**.

Theorem (Radius of convergence)**.** *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series. Then there exists some $R \ge 0$ such that:*

1. *For any R_0 such that $0 \le R_0 < R$, the power series $f(x)$ converges uniformly on the domain $|x| \le R_0$.*
2. *Therefore, $f(x)$ converges pointwise (but not necessarily uniformly) on the domain $|x| < R$.*
3. *When $|x| > R$, $f(x)$ diverges; and anything can happen on the boundary $|x| = R$.*

Furthermore, if $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ *exists, then* $R = \frac{1}{\rho}$*, where we define* $R = \infty$ *when* $\rho = 0$*.*

A careful application of uniform convergence then shows that on the domain $|x| < R$, $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is both integrable on any closed subinterval and differentiable; and furthermore,

$$\int_0^x f(t)\,dt = \int_0^x \sum_{n=0}^{\infty} a_n t^n\,dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n\,dt = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} = \sum_{k=1}^{\infty} \frac{a_{k-1}x^k}{k}, \tag{26}$$

$$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{d}{dx}\left(a_n x^n\right) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{k=0}^{\infty} (k+1)a_{k+1}x^k, \tag{27}$$

when $|x| < R$. In other words, as long as we stay strictly inside the radius of convergence, term-by-term integration and differentiation work correctly and we can push an indefinite integral or derivative past the infinite sum.