Sample Exam 2
Math 129B, Spring 2012

This exam covered through section 6.6 of the text, and not 6.8. Otherwise, it covers roughly the same material as ours will. Our exam may also be a bit longer (this was a 50 minute exam).

1. (15 points) State the Contraction Theorem.

2. (15 points) Let V be a vector space with basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and let $T : V \to \mathbb{M}(2,2)$ be a linear function such that

$$T(\mathbf{v}_1) = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}, \qquad T(\mathbf{v}_2) = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}, \qquad T(\mathbf{v}_3) = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}.$$

Finally, let $\mathbf{x} = \mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3$, and recall that $B = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$.

(a) Compute $T(\mathbf{x})$. Show all your work.

(b) Compute $[T(\mathbf{x})]_B$. Show all your work.

For questions 3–5, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer as specifically as possible. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

3. (12 points) Let V be a vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq V$ be linearly independent. It is possible that $2(\mathbf{v}_1 - \mathbf{v}_2) + 3(\mathbf{v}_2 - \mathbf{v}_3) + (\mathbf{v}_3 - \mathbf{v}_1) = \mathbf{0}$.

4. (12 points) Let $\{f, g, h\}$ be a linearly independent subset of $\mathbb{F}(\mathbb{R})$. It is possible that $3f(0) - 2g(0) + h(0) = 0$.

5. (12 points) Let V be a vector space, and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be vectors in V such that $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Then it is possible that there exist vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5$ in V such that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$ is linearly independent.

6. (16 points) **PROOF QUESTION.** Let V be a vector space, and let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be vectors in V such that

$$\mathbf{v}_1 = \mathbf{u}_1 + \mathbf{u}_3, \qquad \mathbf{v}_2 = 2\mathbf{u}_2 + 3\mathbf{u}_3, \qquad \mathbf{v}_3 = -2\mathbf{u}_1 + 2\mathbf{u}_2 + \mathbf{u}_3.$$

Exactly **one** of the following statements is always true:

- If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ spans V, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans V.
- If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans V, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ spans V.

Circle the statement that is **always** true, and prove it. (Hint: Your proof should not involve much computation.)

7. (18 points) **PROOF QUESTION.** Let V be a vector space, and let W be a subspace of V such that $\dim V = 5$ and $\dim W = 2$. Prove that there exist vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in V such that $\{\mathbf{v}_1, \mathbf{v}_2\}$ spans W and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent.