Math 129B, problem set 11
Outline due: Wed May 09
Due: Mon May 14
Last revision due: TBA

Problems to be done, but not turned in: (8.4) 1, 5, 7; and:

- Let Q be an $n \times n$ matrix whose columns are $\mathbf{u}_1, \dots, \mathbf{u}_n$, in that order. Carefully prove that Q is orthogonal if and only if $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is an orthonormal basis for $\mathbb{R}^n$. (Suggestion: Recall that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^t \mathbf{y}$.)

Problems to be turned in:

1. *The Orthonormal Expansion Theorem:* Let V be an inner product space of dimension n, and let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be an orthonormal subset of V. Prove that there exist vectors $\mathbf{x}_{k+1}, \dots, \mathbf{x}_n$ such that $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{x}_{k+1}, \dots, \mathbf{x}_n\}$ is an orthonormal basis for V. (Suggestion: Use (4.4) 18.)

2. Let $A = \begin{bmatrix} -1 & 0 & -5 & 2 \\ 0 & -1 & -2 & 5 \\ -5 & -2 & -1 & 0 \\ 2 & 5 & 0 & -1 \end{bmatrix}$. Find an orthogonal matrix Q such that $Q^{-1}AQ = Q^tAQ$ is diagonal. You may take it as given that the characteristic polynomial of A is $(x-2)(x+4)(x-6)(x+8)$; you may also want to use a calculator or computer to help with the (mildly tedious) row-reduction.

3. Let A be an $n \times k$ matrix, and let $X = A^tA$. Note that X is a $k \times k$ matrix. This problem works out some technical details that we will need to study the singular value decomposition of A. In the following, in $\mathbb{R}^n$ or $\mathbb{R}^k$, you may use either $\mathbf{x} \cdot \mathbf{y}$ or $\langle \mathbf{x}, \mathbf{y} \rangle$ to denote the inner product of $\mathbf{x}$ and $\mathbf{y}$.

   (a) Prove that there exists an orthonormal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ for $\mathbb{R}^k$ such that each $\mathbf{v}_i$ is an eigenvector of X.

   (b) For $\mathbf{v} \in \mathbb{R}^k$, describe a natural way to find $\mathbf{w} \in \mathbb{R}^n$ such that $\|\mathbf{w}\|^2 = \mathbf{v} \cdot X\mathbf{v}$. (Suggestion: Use $\mathbf{w} \cdot \mathbf{w} = \mathbf{w}^t\mathbf{w}$.)

   (c) For $1 \le i \le k$, let λ_i be the eigenvalue of X associated with $\mathbf{v}_i$ (from part (a)). Use part (b) to prove that $\lambda_i \ge 0$ and that $\lambda_i = 0$ if and only if $A\mathbf{v}_i = \mathbf{0}$.

   (d) Now, by reordering $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ if necessary, we may assume that $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_k \ge 0$. Then, for $1 \le i \le k$, let $\sigma_i = \sqrt{\lambda_i}$. (Note that σ_i is a real number, since $\lambda_i \ge 0$.) Let r be the largest integer such that $\lambda_r > 0$; i.e., pick r so that

   $$\lambda_1 \ge \dots \ge \lambda_r > 0 = \lambda_{r+1} = \dots = \lambda_k.$$

   Finally, for $1 \le i \le r$, let $\mathbf{u}_i = \dfrac{1}{\sigma_i} A\mathbf{v}_i$. (Note that for $1 \le i \le r$, $\sigma_i = \sqrt{\lambda_i} > 0$.) Use part (b) to prove that $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ is an orthonormal subset of $\mathbb{R}^n$.