**Math 129B, problem set 09**
**Outline due: Wed Apr 18**
**Due: Mon Apr 30**
**Last revision due: Mon May 14**

**Important terms and symbols:**

**multiplicity** Let $A$ be an $n \times n$ matrix, and suppose that $c$ is an eigenvalue of $A$. We define the *multiplicity* of $c$ to be the largest integer $m$ such that $(\lambda - c)^m$ divides the characteristic polynomial of $A$.

**Problems to be done, but not turned in:** (8.2) 1, 3, 9, 12, 15; (8.3) 2, 5, 7.

**Problems to be turned in:**

1. (a) For $\mu > 0$, prove that $\{\cos(\mu x), \sin(\mu x)\}$ is linearly independent. (Suggestion: Use the definition.)

   (b) For $\lambda > 0$, prove that $\left\{e^{\lambda x}, e^{-\lambda x}\right\}$ is linearly independent.

   (c) Let $\{\lambda_1, \dots, \lambda_\ell\}$ and $\{\mu_1, \dots, \mu_m\}$ be sets of distinct positive real numbers. (Note that we may have $\lambda_i = \mu_j$.) Prove that

   $$\left\{e^{\lambda_1 x}, e^{-\lambda_1 x}, \dots, e^{\lambda_\ell x}, e^{-\lambda_\ell x}, \cos(\mu_1 x), \sin(\mu_1 x), \dots, \cos(\mu_m x), \sin(\mu_m x)\right\}$$

   is linearly independent.

2. Fix a constant $c \in \mathbb{R}$.

   (a) Let $A$ be an $n \times n$ matrix, and suppose that $k = \dim E_A(c) > 0$. Prove that there exists an invertible matrix $P$ such that columns $1, \dots, k$ of $P^{-1}AP$ are $c\mathbf{e}_1, \dots, c\mathbf{e}_k$, respectively.

   (b) Let $A$ be an $n \times n$ matrix, and suppose that $c$ is an eigenvalue of $A$. Use part (a) to prove that $\dim E_A(c) \leq \text{multiplicity}(c)$.

3. Consider the matrices

   $$A_1 = \begin{bmatrix} -5 & 0 & -6 \\ 3 & 1 & 3 \\ 3 & 0 & 4 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & -1 & -3 \\ -2 & 3 & 6 \\ 4 & -1 & -5 \end{bmatrix}.$$

   You may take it as given that the characteristic polynomials of $A_1$ and $A_2$ are both equal to $(x-1)^2(x+2)$. For each of the matrices $A_i$, either find a matrix $P$ such that $P^{-1}A_iP$ is diagonal, or prove that no such $P$ exists.

(cont. on other side)

4. Suppose $A$ is an $n \times n$ matrix such that $A(A - 3I_n) = 0$. (Note that $(A - 3I_n)A = A^2 - 3A = A(A - 3I_n) = 0$ as well.)

   (a) Prove that every $\mathbf{v} \in \mathbb{R}^n$ is a linear combination of $A\mathbf{v}$ and $(A\mathbf{v} - 3\mathbf{v})$.

   (b) Prove that if $\lambda$ is an eigenvalue of $A$, then either $\lambda = 0$ or $\lambda = 3$.

   (c) Prove that for $\mathbf{v} \in \mathbb{R}^n$, $(A\mathbf{v} - 3\mathbf{v}) \in E_A(0)$ and $A\mathbf{v} \in E_A(3)$. (Suggestion: Think of $E_A(0)$ and $E_A(3)$ as nullspaces.)

   (d) Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis for $E_A(0)$, and let $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ be a basis for $E_A(3)$. Prove that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ spans $\mathbb{R}^n$.

   (e) Prove that $A$ is diagonalizable, and describe what a diagonal matrix similar to $A$ will look like.

5. Let $V$ be a vector space such that $\dim V = 4$, and let $T : V \to V$ and $L : V \to V$ be linear. Suppose there exist nonzero vectors $\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ such that $\mathbf{w}, \mathbf{x} \in E_T(3)$, $\mathbf{y}, \mathbf{z} \in E_T(-7)$, $\mathbf{w}, \mathbf{y} \in E_L(11)$, and $\mathbf{x}, \mathbf{z} \in E_L(-13)$. Prove that there exists a basis $B$ for $V$ such that $[T]_{B,B}$ and $[L]_{B,B}$ are diagonal. (In particular, describe $[T]_{B,B}$ and $[L]_{B,B}$.)