Math 129B, problem set 07
Outline due: Wed Apr 04
Due: Mon Apr 09
Last revision due: Mon May 14

Important terms and symbols:

matrix of a linear function The matrix of the linear function T relative to the bases B (domain) and B' (range) is denoted by $[T]_{B,B'}$. By definition, this is the matrix $A = [T]_{B,B'}$ such that

$$A(B\text{-coordinates of } \mathbf{v}) = B'\text{-coordinates of } T(\mathbf{v}).$$

More precisely, if $B = \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$, then we have

$$[T]_{B,B'} = \Big[[T(\mathbf{u}_1)]_{B'} \ \cdots \ [T(\mathbf{u}_k)]_{B'} \Big].$$

That is, the ith column of $[T]_{B,B'}$ is $[T(\mathbf{u}_i)]_{B'}$.

Problems to be done, but not turned in: (6.7) 1, 9; (6.8) 2, 4; (6.3) 1, 11.

Problems to be turned in:

1. Let $T : V \to W$ be linear, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for V. Prove that $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is a basis for W if and only if T is an isomorphism.

2. (a) Does there exist a linear function $S : \mathbb{R}^3 \to \mathbb{P}_4$ such that S is one-to-one? If so, write down a formula for one such S, and prove that your S is one-to-one; if not, prove that no such S can exist.

   (b) Let
   $$V = \operatorname{span}\left\{1, \cos x, \sin x, e^x, e^{2x}\right\} \subseteq \mathbb{F}(\mathbb{R}).$$
   (You may assume that $\left\{1, \cos x, \sin x, e^x, e^{2x}\right\}$ is linearly independent.) Does there exist a linear function $T : V \to \mathbb{M}(3,2)$ such that T is onto? If so, write down a formula for one such T, and prove that your T is onto; if not, prove that no such T can exist.

   (c) Does there exist a linear function $L : \mathbb{P}_4 \to \mathbb{M}(3,2)$ such that $\operatorname{rank} L = 3$? If so, write down a formula for one such L, and prove that $\operatorname{rank} L = 3$; if not, prove that no such L can exist.

3. Let V and W be vector spaces, and let U be a subspace of W such that $\dim V = 5$, $\dim W = 7$, and $\dim U = 3$. Prove that there exists a linear function $T : V \to W$ such that $\operatorname{nullity} T = 1$ and $\dim(U \cap (\operatorname{im} T)) = 2$. (Note that as part of your proof, you must prove that $\operatorname{nullity} T$ and $\dim(U \cap (\operatorname{im} T))$ are what you claim they are.)

(cont. on other side)

4. Let $I : \mathbb{P}_2 \to \mathbb{P}_3$ be defined by the formula

$$I(p(x)) = \int_0^x p(t)\,dt.$$

In other words, I is indefinite integration, choosing the constant $C = 0$.

(a) Recall that $B_1 = \{1, x, x^2\}$ is an ordered basis for $\mathbb{P}_2$ and $B_2 = \{1, x, x^2, x^3\}$ is an ordered basis for $\mathbb{P}_3$. Find $[I]_{B_1,B_2}$, the matrix of I relative to B_1 and B_2.

(b) It can be shown (i.e., take it as given) that $B_3 = \{1, x+1, (x+1)^2\}$ is an ordered basis for $\mathbb{P}_2$. Find $[I]_{B_3,B_2}$, the matrix of I relative to B_3 and B_2.

(c) It can be shown (i.e., take it as given) that $B_4 = \{1, x+1, (x+1)^2, (x+1)^3\}$ is an ordered basis for $\mathbb{P}_3$. Find $[I]_{B_1,B_4}$, the matrix of I relative to B_1 and B_4.

5. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be a basis for $\mathbb{R}^k$, and let $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ be k vectors in $\mathbb{R}^n$. Recall that that the Whatever Theorem implies that there exists a unique linear $T : \mathbb{R}^k \to \mathbb{R}^n$ such that $T(\mathbf{u}_i) = \mathbf{w}_i$, and that Thm. 6.10 implies that $T = \mu_D$ for some $n \times k$ matrix D. The point of this problem is to find an explicit formula for D.

(a) Describe the matrix A such that $A\mathbf{e}_i = \mathbf{u}_i$ for $1 \le i \le k$, with justification. What size is A?

(b) Describe the matrix B such that $B\mathbf{u}_i = \mathbf{e}_i$ for $1 \le i \le k$, with justification. What size is B?

(c) Describe the matrix C such that $C\mathbf{e}_i = \mathbf{w}_i$ for $1 \le i \le k$, where $\mathbf{e}_i \in \mathbb{R}^k$. What size is C?

(d) Describe the matrix D such that $D\mathbf{u}_i = \mathbf{w}_i$ for $1 \le i \le k$, with justification. What size is D?