Math 129b, problem set 06
Outline due: Wed Mar 14
Due: Wed Mar 21
Last revision due: Wed Apr 18

Problems to be done, but not turned in: (6.2) 1, 5, 10, 15, 20; (6.6) 1, 5, 7.

Problems to be turned in:

1. (6.2) 12.

2. (6.2) 13.

3. Recall that $\mathbb{P}_4$ is the vector space of all polynomials of degree ≤ 4; in particular, an arbitrary element of $\mathbb{P}_4$ has the form $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ ($a_i \in \mathbb{R}$). Let $T : \mathbb{P}_4 \to \mathbb{P}_4$ be defined by the formula

$$T(p(x)) = p(x) - p(2)$$

for all $p(x) \in \mathbb{P}_4$.

   (a) Prove that T is linear.

   (b) Find a basis for $\ker T$, a basis for $\operatorname{im} T$, the rank of T, and the nullity of T.

4. Let $T : V \to W$ be linear. For $\mathbf{b} \in W$, let $S_{\mathbf{b}}$ be the solution set to the equation $T(\mathbf{x}) = \mathbf{b}$ (i.e., let $S_{\mathbf{b}} = \{\mathbf{x} \in V \mid T(\mathbf{x}) = \mathbf{b}\}$). Prove that if $\mathbf{x}_0 \in V$ is one solution to the equation $T(\mathbf{x}) = \mathbf{b}$, then

$$S_{\mathbf{b}} = x_0 + \ker T = \{\mathbf{x}_0 + \mathbf{v} | \, \mathbf{v} \in \ker T\}.$$

(Make sure you do the set containment in both directions.)

5. Recall that $\mathbb{P}$ is the vector space of all polynomials (of any degree). Define linear maps $D : \mathbb{P} \to \mathbb{P}$ and $I : \mathbb{P} \to \mathbb{P}$ by the formulas

$$D(p(x)) = p'(x), \qquad I(p(x)) = \int_0^x p(t)\, dt.$$

In other words, D is differentiation, and I is indefinite integration, choosing the constant $C = 0$.

   (a) Give as precise a description as possible of exactly which polynomials are in $\ker D$ and $\operatorname{im} D$. (I.e., your description should let a reader know which polynomials are in $\ker D$ and $\operatorname{im} D$ without requiring the reader to do any computation.) Is D one-to-one? Is D onto?

   (b) Give as precise a description as possible of exactly which polynomials are in $\ker I$ and $\operatorname{im} I$. Is I one-to-one? Is I onto?

   (c) Is $D \circ I = \mathrm{id}_{\mathbb{P}}$? Is $I \circ D = \mathrm{id}_{\mathbb{P}}$? Are D and I inverses?

(cont. on other side)

6. Let V be a finite-dimensional vector space, and let $T : V \to V$ be linear. Prove that exactly one of the following is true:

   - The equation $T(\mathbf{x}) = \mathbf{b}$ has a solution $\mathbf{x} \in V$ for all $\mathbf{b} \in V$.
   - $\text{nullity}(T) > 0$.

   (Aside: This theorem is known as the *Fredholm Alternative*.)