Math 129B, problem set 05
Outline due: Wed Mar 07
Due: Mon Mar 12
Last revision due: Wed Apr 18

Problems to be done, but not turned in: (3.6) 1, 5, 7, 9; (6.1) 4, 9, 13, 15, 21, 23.

Problems to be turned in:

1. Let $V = \mathbb{P}_2$ (the vector space of all polynomial functions of degree ≤ 2), and let $p_1(x), p_2(x), p_3(x)$ be elements of V such that
$$p_1(-1) = p_2(-1) = p_3(-1) = 0.$$
   (a) Is it possible that $\{p_1, p_2, p_3\}$ spans V? Justify your answer.
   (b) Is it possible that $\{p_1, p_2, p_3\}$ is linearly independent? Justify your answer.

2. First read problem 16 on p. 134; the answer to the problem is essentially given in the "suggestion".

   Now let V be a finite-dimensional vector space, and let S, U, and W be subspaces of V. Explain how to choose bases for S, U, W, $S \cap U$, $S \cap W$, $U \cap W$, and $S \cap U \cap W$ so that:

   - The basis you choose for $S \cap U \cap W$ is a subset of the basis you choose for S, and so on, for all possible subspace containments; and
   - The intersection of the basis you choose for S and the basis you choose for U is the basis you choose for $S \cap U$, and so on, for all possible subspace intersections.

3. Let B be a ordered basis for an n-dimensional vector space V, and define $C_B : V \to \mathbb{R}^n$ by $C_B(\mathbf{v}) = [\mathbf{v}]_B$. Prove that C_B is linear. (See (3.6) 8.)

4. Let $V = \mathbb{D}^\infty([0, 1])$, the space of infinitely differentiable functions on the interval $[0, 1]$, let $\mathbb{N}$ be the set of natural numbers (positive integers), and recall that $\mathbb{F}(\mathbb{N})$ is the vector space of all functions $f : \mathbb{N} \to \mathbb{R}$.

   Let $T : V \to \mathbb{F}(\mathbb{N})$ be defined by the formula
$$(T(f))(n) = \int_0^1 f(x) \cos(2\pi n x)\, dx$$
   for all $n \in \mathbb{N}$. Prove that T is linear.

   Note: T is essentially part of what is known as *the Fourier series transform of f*, though you don't need to know or use that for this problem.

(cont. on other side)

5. Fix $a \in \mathbb{R}$, and let

$$W_a = \left\{ f \in \mathbb{F}(\mathbb{R}) \,\middle|\, \lim_{x \to a} f(x) \text{ exists} \right\}.$$

(a) Prove that W_a is a subspace of $\mathbb{F}(\mathbb{R})$. (This proof mostly involves citing appropriate facts from calculus. You do not need to prove those facts; just state precisely which facts from calculus you need at the appropriate points in time.)

(b) Let $L_a : W \to \mathbb{R}$ be given by

$$L_a(f) = \lim_{x \to a} f(x)$$

for all $f \in W$. Prove that L_a is linear. (Again, this proof mostly involves citing appropriate facts from calculus. You do not need to prove those facts; just state precisely which facts you need at the appropriate points in time.)

6. Prove the *SPAM Lemma:* If $T : V \to W$ is linear and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ spans V, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ spans $\operatorname{im} T$.