The Long Theorems
Linear algebra II (Math 129B)

In this course and in 129A, we have encountered the following TFAE theorems about matrices and linear functions. Let A be an $n \times k$ matrix, and let $T : \mathbb{R}^k \to \mathbb{R}^n$ be the linear function defined by $T(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^k$, i.e., let $T = \mu_A$.

When A is "fat" (i.e., $k \geq n$), we have seen (both in this course and in 129A) that:

Theorem (The Fat Matrix Theorem)**.** *Let A be an $n \times k$ matrix, and let $T : \mathbb{R}^k \to \mathbb{R}^n$ be the linear function defined by $T(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^k$. Then the following are equivalent:*

1. *The columns of A span $\mathbb{R}^n$.*
2. *For every $\mathbf{b} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ has at least one solution $\mathbf{x} \in \mathbb{R}^k$. (I.e., the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.)*
3. $\operatorname{rank} A$ *is equal to the height of the matrix A.*
4. $\operatorname{rref} A$ *has no zero rows.*
5. *T is onto.*
6. $\operatorname{im} T = \mathbb{R}^n$.
7. $\operatorname{rank} T = n$.

The Fat Matrix Theorem also holds for "tall" matrices ($n > k$), but in that case, all of the conditions are always false.

Similarly, when A is tall ($n \geq k$), we have seen that:

Theorem (The Tall Matrix Theorem)**.** *Let A be an $n \times k$ matrix, and let $T : \mathbb{R}^k \to \mathbb{R}^n$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^k$. Then the following are equivalent:*

1. *The columns of A are linearly independent.*
2. *The only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.*
3. *For every $\mathbf{b} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ has at most one solution $\mathbf{x} \in \mathbb{R}^k$.*
4. *The columns of* $\operatorname{rref} A$ *are $\mathbf{e}_1, \ldots, \mathbf{e}_k$.*
5. *Every column of A is a pivot column.*
6. *Every column of* $\operatorname{rref} A$ *is a pivot column.*
7. $\operatorname{rank} A = k$ *(the width of the matrix A).*
8. $\operatorname{nullity} A = 0$.
9. $\operatorname{Null} A$ *is the zero subspace of $\mathbb{R}^k$.*
10. *T is one-to-one.*
11. $\operatorname{rank} T = k$.
12. $\operatorname{nullity} T = 0$.
13. $\ker T$ *is the zero subspace of $\mathbb{R}^k$.*

Again, the Tall Matrix Theorem also holds for fat matrices ($n < k$), but in that case, all of the conditions are always false.

(cont.)

When A is square ($n = k$), or equivalently, when the domain and range of T have the same finite dimension, we combine the above two theorems and a few other things we have seen to get the following theorem. Note that *this only works when A is square.*

Theorem (The Long Theorem)**.** *Let A be an $n \times n$ matrix, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$. Then the following are equivalent:*

1. *A is invertible.*
2. *There exists an $n \times n$ matrix B such that $BA = I_n$.*
3. *There exists an $n \times n$ matrix C such that $AC = I_n$.*
4. *Every column of A is a pivot column.*
5. *Every column of* $\operatorname{rref} A$ *is a pivot column.*
6. $\operatorname{rref} A$ *has no zero rows.*
7. $\operatorname{rref} A = I_n$.
8. *The only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.*
9. *For every $\mathbf{b} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ has at least one solution $\mathbf{x} \in \mathbb{R}^n$. (I.e., the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.)*
10. *For every $\mathbf{b} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ has at most one solution $\mathbf{x} \in \mathbb{R}^n$.*
11. $\operatorname{rank} A = n$.
12. $\operatorname{nullity} A = 0$.
13. $\operatorname{Null} A$ *is the zero subspace of* $\mathbb{R}^n$.
14. *The columns of A span $\mathbb{R}^n$.*
15. *The columns of A are linearly independent.*
16. *The columns of A are a basis for $\mathbb{R}^n$.*
17. *T is onto.*
18. *T is one-to-one.*
19. *T is invertible.*
20. $\operatorname{im} T = \mathbb{R}^n$.
21. $\operatorname{rank} T = n$.
22. $\operatorname{nullity} T = 0$.
23. $\ker T$ *is the zero subspace of* $\mathbb{R}^n$.
24. *A is a product of elementary matrices.*
25. $\det A \neq 0$.
26. 0 *is not an eigenvalue of A.*
27. 0 *is not an eigenvalue of T.*

Exercise: Try to prove the Long Theorem. More tractably, take any two parts of the Long Theorem and try to prove that one implies the other.