The contraction algorithm
Math 129B

The contraction algorithm states:

Theorem. *For* $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^\ell$*, let* $S = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$*, and let* M *be the* $\ell \times n$ *matrix whose columns are* $\mathbf{v}_1, \ldots, \mathbf{v}_n$*. Then the vectors corresponding to the leading variables in the reduced row-echelon form of* M *form a basis for* S*.*

Proof. First, we claim that, **without loss of generality**, by renumbering $\mathbf{v}_1, \ldots, \mathbf{v}_n$, we may assume that the RREF of M has the form

$$\begin{bmatrix} 1 & & 0 & * & \cdots & * \\ & \ddots & & \vdots & \ddots & \vdots \\ 0 & & 1 & * & \cdots & * \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{*}$$

To verify the claim, first find the RREF of M. Once we know which columns will be the leading columns, before doing Gaussian reduction, renumber the original $\mathbf{v}_1, \ldots, \mathbf{v}_n$ to put the leading columns of M on the left side of M in the correct order, and perform exactly the same reduction as before. Since row operations have the same effect on columns, no matter what order they are in, the result will be a RREF of the form shown above.

So now, given the WLOG, let k be the number of leading variables in the RREF of M, let $r_1, \ldots, r_k$ and $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be the leading variables and corresponding vectors, and let $r_{k+1}, \ldots, r_n$ and $\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n$ be the free variables and corresponding vectors. We need to show that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ spans S and is linearly independent.

To show that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ spans S, it is enough to show that each of the vectors $\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. However, for i such that $k+1 \leq i \leq n$, if we pick the free variable $r_i = -1$ and the other free variables $r_{k+1} = \cdots = r_n = 0$, there exists a unique choice of $r_1, \ldots, r_k$ solving the homogeneous equations (*). Going back to the original equations M, we see that this solution gives the identity

$$r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k - \mathbf{v}_i = \mathbf{0}.$$

It follows that $\mathbf{v}_i$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

To show that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, suppose that

$$r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k = \mathbf{0}$$

for some $r_1, \ldots, r_k \in \mathbb{R}$. In that case, we also have

$$r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k + 0\mathbf{v}_{k+1} + \cdots + 0\mathbf{v}_n = \mathbf{0}.$$

In other words, $r_1, \ldots, r_k$ and $r_{k+1} = \cdots = r_n = 0$ is a solution to the homogeneous equations (*). However, since the equations (*) express each of $r_1, \ldots, r_k$ in a solution as a linear combination of $r_{k+1}, \ldots, r_n$, we must have $r_1 = \cdots = r_k = 0$. The theorem follows. $\square$