Review of span and linear independence
Linear algebra (Math 129A)

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be vectors in $\mathcal{R}^n$. The fundamental definitions are:

Definition. The *span* of $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is the set of all linear combinations of $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$. In other words, the span of $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is

$$\operatorname{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} = \{a_1\mathbf{u}_1 + \cdots + a_k\mathbf{u}_k \mid a_i \in \mathcal{R}\}.$$

Definition. If, for some $c_1, \ldots, c_k \in \mathcal{R}$ with not all $c_i = 0$, we have

$$c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k = \mathbf{0}, \qquad (*)$$

then we say that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is *linearly dependent.* If the only solution to (*) is $c_1 = \cdots = c_k = 0$, then we say that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is *linearly independent.*

Which sets of vectors span $\mathcal{R}^n$/are linearly independent? Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be vectors in $\mathcal{R}^n$, and let A be the $n \times k$ matrix $[\mathbf{u}_1 \ \cdots \ \mathbf{u}_k]$, i.e., the matrix whose columns are $\mathbf{u}_1, \ldots, \mathbf{u}_k$. Among other things, the following theorems (Thms. 1.5 and 1.7, respectively) give tests for determining if $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ spans $\mathcal{R}^n$ and determining if $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is linearly independent. (Actually, these tests are really just a single test: finding the rank of A.)

Theorem (Fat Matrix Theorem). *For an $n \times k$ matrix A, the following are equivalent:*

1. *The columns of A span $\mathcal{R}^n$.*
2. *For every $\mathbf{b} \in \mathcal{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has either one solution or infinitely many solutions.*
3. $\operatorname{rank}(A) = n$.
4. *RREF(A) has no zero rows.*

We call this the Fat Matrix Theorem because for the conditions to be true, we must have $k \geq n$ (i.e., the matrix A must be "fat").

Theorem (Tall Matrix Theorem). *For an $n \times k$ matrix A, the following are equivalent:*

1. *The columns of A are linearly independent.*
2. *The only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.*
3. *For every $\mathbf{b} \in \mathcal{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has either no solutions or one solution.*
4. $\operatorname{rank}(A) = k$.
5. *Every column of RREF(A) is a pivot column.*

We call this the Tall Matrix Theorem because for the conditions to be true, we must have $n \geq k$ (i.e., the matrix A must be "tall").

Enlarging or shrinking spanning sets. Here, we start to see how the ideas of span and linear independence complement each other.

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be vectors in $\mathcal{R}^n$. In Thm. 1.8, we see that:

Theorem. *The vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly dependent precisely if one of the following conditions is true:*

1. *Either $\mathbf{u}_1 = \mathbf{0}$, or*
2. *Some $\mathbf{u}_i$ $(2 \leq i \leq k)$ is a linear combination of the previous vectors.*

Combining this with part (c) of Thm. 1.6, we see that:

Theorem. *Let S be a finite set of vectors in $\mathcal{R}^n$, and let $V = \operatorname{Span} S$. Then V can be spanned by a smaller subset of S if and only if S is linearly dependent.*

Proof. If S is linearly dependent, then either some vector in S is equal to $\mathbf{0}$ or at least one vector $\mathbf{z} \in S$ is a linear combination of the others. By Thm. 1.6(c), we can remove $\mathbf{z}$ from S and obtain a smaller set of vectors with the same span.

Conversely, suppose we can remove a vector $\mathbf{z}$ from S and obtain a smaller set of vectors with the same span. In that case, by Thm. 1.6, $\mathbf{z}$ is a linear combination of the other vectors in S, so by Thm. 1.8, S is linearly dependent. □

The Span-Independence Theorem. Another key relationship between spanning and linear independence is Thm. 1.9, whose importance will become clearer later.

Theorem (Span-Independence Theorem). *Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be vectors in $\mathcal{R}^n$, and let $V = \operatorname{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$. Every subset of V containing more than k vectors is linearly dependent.*

In other words, put in terms of linear independence:

Theorem. *Let V be a subset of R^n. Any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ that spans V is at least as large as any linearly independent subset $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ of V.*

The point is, we do not assume that the $\mathbf{v}$'s have any direct relation to the $\mathbf{w}$'s (e.g., $\mathbf{v}_1$ need not be $\mathbf{w}_1$, etc.), but we still know that there have to be more $\mathbf{v}$'s than $\mathbf{w}$'s.