Sample final, Spring 2002

1. Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be vectors in a subspace V of $\mathcal{R}^n$.

(a) (10 points) Define what it means for $\mathbf{u}_1, \ldots, \mathbf{u}_k$ to be linearly independent.

(b) (10 points) Define what it means for $\mathbf{u}_1, \ldots, \mathbf{u}_k$ to be a basis for V, and define the dimension of V.

2. (14 points) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -3 & -4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & -5 \\ -1 & 0 \end{bmatrix}$.

(a) Exactly one of the products AB, BA is defined. Calculate that product.

(b) Calculate A^{-1}, if it exists, or explain how you know that it does not exist.

3. (14 points) Find the solution set to

$$\begin{aligned} 2x_2 \qquad\quad + 2x_4 &= 4, \\ -x_1 - 2x_2 + x_3 \qquad &= 2, \\ 2x_1 + x_2 - 2x_3 \qquad &= -1. \end{aligned}$$

If the system is consistent, put your final answer in vector form; if the system is not consistent, explain how you know that the system is not consistent.

4. (20 points) Let

$$A = \begin{bmatrix} 1 & 2 & -4 & 1 & 0 \\ 0 & 1 & -3 & 1 & 1 \\ 3 & 2 & 0 & 0 & -3 \\ -1 & 1 & -5 & 1 & 2 \end{bmatrix}, \qquad \text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis for the column space of A.

(b) Find the dimension of the nullspace of A.

(c) Find **one** specific nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$.

Show all your work, and in each part this question, briefly **EXPLAIN** (in a phrase or sentence) how your answer was obtained.

5. (14 points) Let $W = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix} \right\}$. Find an orthogonal basis for W.

Show all your work.

6. (20 points) Let $A = \begin{bmatrix} -1 & -3 & -6 \\ -6 & -4 & -12 \\ 3 & 3 & 8 \end{bmatrix}$. A computation shows that the characteristic polynomial of A is $-(t+1)(t-2)^2$. (This is given; do not spend time checking it.)

Determine if A is diagonalizable. If A is diagonalizable, find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$; if A is not diagonalizable, explain how you know that A is not diagonalizable.

7. (T/F) (8 points) Let A be a 4×4 matrix such that 3 is an eigenvalue of A. It is possible that the nullspace of $(A - 3I_4)$ is the zero subspace.

8. (T/F) (8 points) Let V be a 2-dimensional subspace of $\mathcal{R}^4$, and let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be nonzero vectors in V. Then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ must be linearly independent.

9. (T/F) (8 points) There exist 3×3 matrices A and B such that $\det A = 2$, $\det B = -4$, and AB is not invertible.

10. (T/F) (8 points) Let W be a nonzero subspace of $\mathcal{R}^5$. If $\mathbf{v}$ is a vector in $\mathcal{R}^5$, and $\mathbf{v} = \mathbf{x} + \mathbf{y}$, where $\mathbf{x} \in W$ and $\mathbf{y} \in W^{\perp}$, then $\mathbf{x}$ is the vector of W that is closest to $\mathbf{v}$.

11. (T/F) (8 points) If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are vectors in $\mathcal{R}^4$, and W is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, then W must be a subspace of $\mathcal{R}^4$.

12. (T/F) (8 points) Let $T : \mathcal{R}^3 \to \mathcal{R}^2$ be a linear transformation, let $\mathbf{b}$ be a vector in $\mathcal{R}^2$, and let W be the set of all $\mathbf{x} \in \mathcal{R}^3$ such that $T(\mathbf{x}) = \mathbf{b}$, i.e., $W = \{\mathbf{x} \in \mathcal{R}^3 \mid T(\mathbf{x}) = \mathbf{b}\}$. Then W must be a subspace of $\mathcal{R}^3$.

13. (T/F) (8 points) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and let B be an invertible 2×2 matrix. Then it must be true that $AB = BA$.

14. (T/F) (8 points) Let $T : \mathcal{R}^2 \to \mathcal{R}^4$ be a linear transformation such that $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -3 \\ 5 \\ 0 \end{bmatrix}$. It is possible that $T(\mathbf{x}) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ for some $x \in \mathcal{R}^2$.

15. Let A be a 5×7 matrix, and let $\mathbf{b}$ be a vector in $\mathcal{R}^5$.

(a) (8 points) Is it possible that $\operatorname{rank} A = 6$? Give an example of such an A, or explain why no such A can exist.

(b) (8 points) Is it possible that the equation $A\mathbf{x} = \mathbf{b}$ has **exactly one** (i.e., at least one, and not more than one) solution $\mathbf{x} \in \mathcal{R}^7$? Give an example of such an A, or explain why no such A can exist.

16. Suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ spans a subspace V of $\mathcal{R}^4$.

(a) (9 points) Let $\mathbf{y}$ be a vector in V. **Must** it be true that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{y}\}$ spans V? Either:

- Explain why $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{y}\}$ must span V; or
- Give a specific example of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{y}\}$ satisfying the above conditions, and explain how you know that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{y}\}$ does not span V in your example.

(b) (9 points) **Must** it be true that $\{\mathbf{u}_1, \mathbf{u}_2\}$ spans V? Eiither:

- Explain why $\{\mathbf{u}_1, \mathbf{u}_2\}$ must span V; or
- Give a specific example of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ satisfying the above conditions, and explain how you know that $\{\mathbf{u}_1, \mathbf{u}_2\}$ does not span V in your example.