

**Sample Math 128A Final Exam**  
**Math 128B, Spring 2011**

1. (13 points) Define what it means for  $(G, +)$  to be an **additive abelian** group. In other words, define what it means for  $G$  to be an abelian group under the operation of  $+$ .

2. (17 points)

(a) Define what it means for  $R$  to be a ring.

(b) Now let  $R$  be a ring. Define what it means for a subset  $S$  of  $R$  to be a **subring** of  $R$ .

(c) Again let  $R$  be a ring. Define what it means for a subset  $A$  of  $R$  to be an ideal of  $R$ .

3. (12 points) Let

$$\alpha = (1\ 7)(2\ 5\ 4\ 8)(3\ 11\ 9\ 6\ 12)$$

$$\beta = (1\ 10\ 7)(2\ 11\ 6\ 3\ 9\ 4)(5\ 12\ 8)$$

Calculate  $\alpha\beta$ ,  $\alpha^{-1}$ , and  $|\alpha|$ . Put all permutation answers in cycle form, and show all your work.

4. (12 points)

(a) List all abelian groups of order  $32 = 2^5$ , up to isomorphism. Show all your work.

(b) Which abelian groups of order 32 do **not** contain any elements of order 8? Briefly **justify** your answer.

For questions 5–10, you are given a statement. If the statement is true, you need only write “True”, though a justification may earn you partial credit if the correct answer is “False”. If the statement is false, write “False”, and justify your answer **as specifically as possible**. (Do not just write “T” or “F”, as you may not receive any credit; write out the entire word “True” or “False”.)

5. (13 points) (**TRUE/FALSE**) If  $G$  is an additive abelian group,  $H$  is a subgroup of  $G$ ,  $a \in G$ , and  $(a+H)$  has order 2 in  $G/H$ , then it must be the case that  $a+a=0$  in  $G$  (where 0 is the identity in  $G$ ).

6. (13 points) (**TRUE/FALSE**) It is possible there exists a group  $G$  of order 520 that has a subgroup isomorphic to  $A_4$  (the alternating group of rank 4).

7. (13 points) (**TRUE/FALSE**) If  $G$  is a cyclic group of order 48, then it is possible that  $G$  has a subgroup isomorphic to  $\mathbf{Z}_4 \oplus \mathbf{Z}_4$ .

8. (13 points) (**TRUE/FALSE**) If  $R$  is a ring, and  $S$  is a subgroup of  $R$  under the addition in  $R$ , then it must be the case that  $S$  is a **subring** of  $R$ .

9. (13 points) (**TRUE/FALSE**) If  $\varphi : G \rightarrow \mathbf{Z}_3 \oplus \mathbf{Z}_3$  is a **surjective** (onto) homomorphism, then it must be the case that there exists a normal subgroup  $N$  of  $G$  such that  $G/N$  is a nontrivial abelian group.

10. (13 points) (**TRUE/FALSE**) If  $H$  is a normal subgroup of the group  $G$ ,  $a, b \in G$ , and  $(aH)(bH)^{-1} \neq H$ , then it is possible that  $a = b$ .

11. (17 points) **PROOF QUESTION.** Let  $G$  be a group of order  $63 = 3^2 \cdot 7$ . Prove that  $G$  contains an element of order 3.

12. (17 points) **PROOF QUESTION.** Let  $n$  be an integer  $\geq 3$ , and let

$$H = \{\alpha \in S_n \mid (\alpha(1) = 1 \text{ and } \alpha(2) = 2) \text{ or } (\alpha(1) = 2 \text{ and } \alpha(2) = 1)\}.$$

Prove that  $H$  is a subgroup of  $S_n$ .

13. (17 points) **PROOF QUESTION.**

- (a) Define what it means for a function  $\varphi : G \rightarrow \overline{G}$  to be onto (i.e., to be surjective).
- (b) Define what it means for a function  $\varphi : G \rightarrow \overline{G}$  to be a homomorphism.
- (c) Now let  $G$  and  $\overline{G}$  be groups, and let  $\varphi : G \rightarrow \overline{G}$  be an onto (surjective) homomorphism. Prove that if  $a \in G$  satisfies  $ax = xa$  for all  $x \in G$ , then  $\varphi(a)$  satisfies  $\varphi(a)\overline{x} = \overline{x}\varphi(a)$  for all  $\overline{x} \in \overline{G}$ .

14. (17 points) **PROOF QUESTION.**

- (a) Give an example of a ring  $R$  in which there exist at least 4 distinct elements  $x \in R$  such that  $x^3 = x$ . (Suggestion: Try  $\mathbf{Z}_n$  for a suitable  $n$ .)
- (b) Prove that if  $R$  is an integral domain, then there exist at most 3 distinct elements  $x \in R$  such that  $x^3 = x$ .