Sample Final Exam
Math 128B, Spring 2021

1. (14 points) Let R and S be rings, and let $\varphi : R \to S$ be a map from R to S.

(a) Define what it means for φ to be a homomorphism.

(b) Define $\ker \varphi$, the kernel of φ.

(c) State the First Isomorphism Theorem for φ.

2. (10 points) Let E and F be fields. Define what it means for E to be an extension of F, and define $[E : F]$, the degree of E over F.

3. (15 points) Suppose F is a field, and E is the splitting field of some $f(x) \in F[x]$ over F. Suppose also that $|\mathrm{Gal}(E/F)| = 120$, and $|\mathrm{Gal}(E/F)|$ contains subgroups H_1, H_2, and H_3 such that:

- $|H_1| = 6$, $|H_2| = 24$, and $|H_3| = 8$; and
- $H_1 \leq H_2$ and $H_3 \lhd \mathrm{Gal}(E/F)$.

For $i = 1, 2, 3$, let $K_i = E_{H_i}$, the fixed field of H_i.

(a) Draw as much of the subfield lattice of E as you can derive from the given information.

(b) For each K_i ($i = 1, 2, 3$), indicate the value of $[K_i : F]$.

(c) Can you be certain that any of the K_i are splitting fields of some $f(x) \in F[x]$ over F? For each such K_i, determine the order of $\mathrm{Gal}(K_i/F)$.

4. (15 points) Let $f(x) = x^4 - 32x + 6 \in \mathbf{Q}[x]$, let $I = \langle f(x) \rangle$, and let $F = \mathbf{Q}[x]/I$.

(a) Explain how you can be certain that F is a field.

(b) Find a basis for F as a vector space over $\mathbf{Q}$.

(c) Express the element $(x^2 + I)(x^3 + I) \in F$ as a linear combination of your chosen basis elements in part (b).

For questions 5–9, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

5. (13 points) **(TRUE/FALSE)** Let $f(x)$ be a nonconstant **monic** polynomial with integer coefficients, and let $\overline{f}(x)$ be the polynomial in $\mathbf{Z}_7[x]$ obtained from $f(x)$ by reducing all of the coefficients of $f(x)$ (mod 7). It must be the case that if $\overline{f}(x)$ is irreducible over $\mathbf{Z}_p$, then $f(x)$ is irreducible over $\mathbf{Q}$.

6. (13 points) **(TRUE/FALSE)** Let D be a Euclidean domain, let p be an irreducible element of D, and let a, b be nonzero elements of D. It is possible that p divides ab, p does not divide a, and p does not divide b.

7. (13 points) **(TRUE/FALSE)** Let $f(x) \in \mathbf{Q}[x]$ be a nonconstant polynomial, and let a be an element of some extension of $\mathbf{Q}$ such that $f(a) = 0$. Then it must be the case that $\mathbf{Q}(a)$ is the splitting field of $f(x)$ over $\mathbf{Q}$.

8. (13 points) **(TRUE/FALSE)** Let S be a **subring** of $\mathbf{R}[x]$ (the ring of real-valued polynomials). Then it must be the case that S is an **ideal** of $\mathbf{R}[x]$.

9. (13 points) **(TRUE/FALSE)** Let E be a field of order 343. Then it must be the case that there exists some $\alpha \in E$ such that $\alpha^i \neq 1$ for $1 \leq i \leq 341$ and $\alpha^{342} = 1$.

10. (17 points) **PROOF QUESTION.** Let R be a commutative ring with unity, let I be an ideal of R, and let c be an element of I such that $c = ab$ for some $a, b \notin I$. Name an element of R/I that is a zero divisor in R/I, and prove your answer.

11. (17 points) **PROOF QUESTION.** Let F be a field of characteristic 0, let E be the splitting field of some $g(x) \in F[x]$ over F, and suppose that $[E : F] = 651 = 3{\cdot}7{\cdot}31$. Suppose also that $\mathrm{Gal}(E/F)$ is abelian. Prove that there exists some $f(x) \in F$ and a subfield K of E containing F such that K is the splitting field of $f(x)$ over F and $[K : F] = 21$.

12. (17 points) **PROOF QUESTION.** Let D be a principal ideal domain, and suppose that I, J are ideals of D such that $I \subseteq J$. Prove that there exists some $d \in J$ such that d divides every element of I.

13. (17 points) **PROOF QUESTION.** Let F be a field, let E be an extension of F, and suppose that:

- There exists $a \in E$ such that $E = F(a)$ and $f(a) = 0$ for some $f(x) \in F[x]$ of degree 55. (Note that we **do not assume** $f(x)$ is irreducible.)
- There exist subfields K and L of E such that $F \subseteq K$, $F \subseteq L$, $[K : F] = 5$, and $[E : L] = 11$.

Prove that E has degree 55 over F and $f(x)$ is irreducible.