Sample Exam 3
Math 128B, Spring 2021

1. (8 points) Consider the polynomial $f(x) = x^3 - 17$ in $\mathbf{Q}[x]$. Find a, b such that $f(x)$ splits in $\mathbf{Q}(a, b)$, and write $f(x)$ as a product of linear factors in $\mathbf{Q}(a, b)$. (You may find the abbreviation $\omega_n = e^{2\pi i/n}$ helpful.)

For questions 2–5, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

2. (12 points) Let a be a positive real number such that $a^5 - 14a - 2 = 0$. It is possible to start with a line segment of length 1, and, using a finite number of operations with straightedge and compass, construct a line segment of length a.

3. (12 points) Let E be a field of order $125 = 5^3$. Then it must be the case that if we consider E as a group under addition, E is isomorphic to the group $\mathbf{Z}_{125}$.

4. (12 points) Let $g(x) \in \mathbf{Q}[x]$ be a polynomial of degree 4 that is irreducible over $\mathbf{Q}$, and let a be a complex number such that $\mathbf{Q}(a) = 0$. If K is an extension of $\mathbf{Q}$ such that $a \notin K$, then it must be the case that $[K(a) : K] = 4$.

5. (12 points) Let $f(x) \in \mathbf{Q}[x]$ be irreducible over $\mathbf{Q}$, and let a be a complex number such that $f(a) = 0$. Then it must be the case that $\mathbf{Q}(a) \approx \mathbf{Q}[x]/\langle f(x) \rangle$.

6. (14 points) **PROOF QUESTION.** Note that $81 = 3^4$. Prove that there exists some $\alpha \in GF(81)$ such that $\alpha^{40} = -1$.

7. (14 points) **PROOF QUESTION.** Let F be a field, and let E be an extension of F such that E has some basis $\{a_1, \ldots, a_{10}\}$ as a vector space over F. Let b be an element of E such that

1. For any nonzero $f(x) \in F[x]$ such that $\deg f \leq 2$, $f(b) \neq 0$; and
2. $F(b) \neq E$.

Prove that $[E : F(b)] = 2$.